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Research Article

The Influence of Curriculum on the Concept of Function: An Empirical Study of Pre-Service Teachers

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ABSTRACT

This article reports on the understanding of the function concept by pre-service mathematics teachers from two countries (Germany and Croatia). We focused on investigating students' concept definition and concept image of the function in relation to their curriculum experiences. Data were collected using a questionnaire in the form of open-ended questions followed by interviews. The results indicate that the curriculum has a great influence on the development of the concept definition and concept image. The curriculum strongly influenced the theoretical background of the function concept and thus the gap between the formal and the personal definition of function. Later and more intensive work with the formal definition of function led to a better development of the function concept in general. The curriculum also had an influence on the range of the concept image developed by the pre-service mathematics teachers, with no proportional dependence in relation to the better developed understanding of the concept of function.

Keywords: concept definition, concept image, function, pre-service mathematics teachers

INTRODUCTION

The concept of function represents the fundamental concept of mathematics as a scientific discipline and the central concept of school mathematics (e.g., Leinhardt et al., 1990; Thompson & Carlson, 2017). Moreover, it is a concept that relates mathematics to the real world. Developing students' ability to understand and work with functions is one of the main goals of mathematics curricula around the world. From the earliest grades, students engage in a variety of activities, including exploring patterns in elementary school, co-varying quantities in middle school, and formally treating functions as mappings between sets in high school (Carlson & Oehrtman, 2005; Cooney et al., 2010; Hatisaru & Erbas, 2017). To support and facilitate student learning, teachers need to have a solid knowledge of functions, including a solid formal definition and appropriate representations of functions that form a coherent concept image of function. Although the concept of function has been studied since 1960 (Dubinsky & Wilson, 2013), there are still some questions that have not been sufficiently explored. For example, how curriculum experiences influence pre-service secondary mathematics teachers' understanding of the concept of function. Such an issue is of interest to us in this study. We asked the same questions to a sample of pre-service secondary mathematics teachers (PSMTs) in two countries, Croatia and Germany. This allowed us to examine their responses both qualitatively and quantitatively, and to make conjectures about the complexity of their understanding of the concept of function based on two different educational backgrounds.

THEORETICAL FRAMING

Mathematical Definitions: The Cornerstone of Mathematical Thinking

In mathematics, definitions play a prominent role in the development of mathematical thinking: they describe objects and concepts, identify basic and essential properties of mathematical objects, support problem solving/proofs, and enable communication of mathematics (Zaslavsky & Shir, 2005). Knowing the formal definition of a mathematical concept does not mean that a person understands the concept in question. Take, for example, the formal definition of a function. Mathematicians throughout history have had difficulties with understanding the concept of function, particularly with the notions of arbitrariness (which is implicit in the definition) and univalence (which is explicit in the definition), and the same difficulties with understanding the concept of function can be observed in contemporary students (Even, 1993).

Concept Image and Concept Definition

An excellent way to test a person's understanding of a mathematical concept is to examine the concept image associated with that concept. Tall and Vinner (1991) introduced concept image as a tool for analyzing the distinction between personal and formal understanding. Concept image is the total cognitive structure associated with a concept. It includes all the ideas that a person has about the concept, such as mental pictures, properties, processes, or metaphors associated with the concept, and arises from the experiences that a person has had throughout his or her life. To understand the formal concept definition, a person interprets it and creates a personal interpretation of the definition. The formal concept definition is usually unambiguous, but the personal interpretation of the definition varies from person to person and depending on the context. A personal concept definition may or may not be consistent with a definition accepted by the broader mathematical community. Tall regards a concept definition as part of the concept image whereas Vinner (1991) makes a distinction between them. Viholainen (2008) argues that the personal interpretation of the concept definition is part of the concept definition image, which is part of the concept image (**Figure 1**). The formal concept definition is linked with the personal interpretation of the concept definition, and through this connection it has a significant impact on the concept image. The personal concept definition can be equal with the personal interpretation of the definition or it may lie outside the concept definition image.

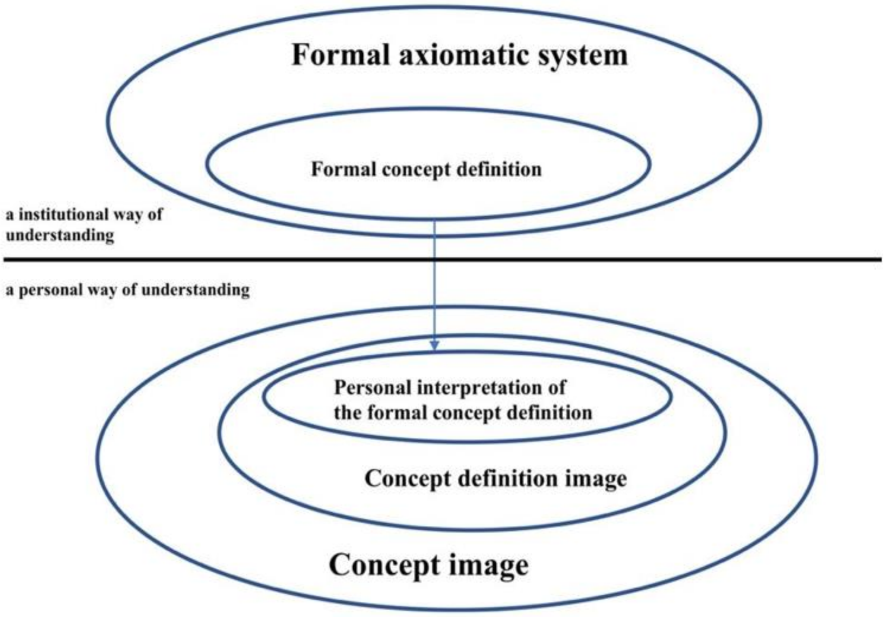


Figure 1. Concept image and concept definition relationship (Viholainen, 2008, p. 234)

Vinner (1991) argues that definitions create problems for learning mathematics because they represent a difference between mathematics as a structured body of knowledge and the cognitive processes of learning mathematics. He further claims that when one hears the name of a concept, it is usually not its definition but a concept image that is recalled in memory. This means that in everyday situations, the definition of concept becomes redundant in the moment when a concept image emerges. In mathematics, definitions do not only help to form concept images but are also important for solving tasks. Tall and Vinner (1981) argue that a definition can either be rote memorized or meaningfully learnt if it is associated with the concept. Viholainen (2008) defines a concept image as coherent if the conception about the concept is clear, all conceptions concerning the concept are connected, there are no internal contradictions, and the concept image does not contain ideas that contradict the formal system of mathematics. Otherwise, the concept image is not coherent. Learning a concept does not occur in one step but over time (Vinner & Deryfus, 1989). Also, students' understanding can be 'partially correct'–this means there is a match between a student's construct and the corresponding formal mathematics (Ron et al., 2010). In this study, the term 'partial understanding' is used when students' stated definition is in part correct but essential elements are missing, or when a concept image is not coherent but shows some consistency with the formal definition.

Concept Definition and Concept Image–Literature Review on Functions

The early introduction of the formal definition of a function in school mathematics is not useful for students; the definition is either ignored or misunderstood (Sierpinska, 1992). In addition, the language associated with the function is also a hindrance; teachers and students use *f(x)* for the value of the function, but also as the name of the function. Studies report that teachers' understanding of the concept of function is often very similar to that of students. Teachers can provide a correct formal definition of a function, but are unable to correctly apply that definition to identify functions and non-functions (Chesler, 2012; Tabach & Nachlieli, 2015). Moreover, in such situations, they emphasize properties of graphs, such as the vertical line test, to distinguish between functions and non-functions. When teachers have a solid concept of functions, their students tend to develop a high level of function content knowledge and when teachers have limitations in their knowledge, students exhibit the same limitations (Hatisaru & Erbas, 2017).

The concept image of function can include different forms of representation: tables, graphs, equations, verbal descriptions. A good understanding of function concept is demonstrated by the ability to use multiple representations and to transfer from one representation to another (NCTM, 2000). Studies report that teachers have a limited repertoire of representations to draw upon when helping students understand functions (e.g., Bannister, 2014). In-service and pre-service teachers prefer symbolic algebraic representations of functions, much like students (Even, 1993; Wilson, 1994). Sintema and Marban (2020) have shown that pre-service teachers who have low self-concept about pedagogical content knowledge have limited knowledge about the representations of functions, inverse functions, and composite functions, so they are unlikely to be confident enough to teach the concept of function at the secondary level. The verbal representation of functions is generally the representation that is used and presented the least in textbooks and the classroom (Bossé et al., 2011). Several studies (e.g., Hadjidemetriou & Williams, 2002; Leinhardt et al., 1990) reported that this type of representation is also the most difficult for students when it comes to translating everyday world mathematics. The difficulty arises from the exchange of the level of abstraction (Nitsch, 2015). Furthermore, Carlson (1998) reported that students in college algebra courses had difficulty constructing a function when it was given as a verbal representation (word problem).

Research Problem

The idea for this study arose from the teaching practice. The first author of this paper was educated in Croatia. While working with PSMTs at a German university, she noticed differences in the ideas that she and her students had about the concept of function. When we examined the literature on the problems and misconceptions related to functions, we also found that the number of studies dealing with PSMTs is much smaller than the number of studies dealing with high school students. Studies on PSMTs rarely include details about their educational background. Such studies examined understanding of the concept and its associated representations without placing them in an adequate sociocultural context. The public image of mathematics, institutions such as schools and universities, and the noosphere of teachers (Rezat & Strasser, 2012) are some of the social and cultural factors that influence which concept is to be learned, how it should be learned, and to what extent. Furthermore, students' experiences with functions are usually based on prototypical examples used by teachers and textbooks as quasi-policy documents (Pepin et al., 2012; Schwarz & Hershkowitz, 1999). Thus, when it comes to PSMTs' understanding, we are interested in the role of the school curriculum and higher education. The purpose of this research is to examine this issue with the concept of function as a central theme, studying two groups of PSMTs whose expectations of the school curriculum and higher education are quite different. The following research questions are formed:

1. What are the differences in the understanding of the concept of function (i.e., concept definition and concept image) between pre-service secondary mathematics teachers with different educational backgrounds?
2. How do diverse educational background affect students' use of the formal and personal definition of a function?

METHODOLOGY

Participants

This study involved 61 pre-service mathematics teachers from two countries, Germany (N=30) and Croatia (N=31). The study was conducted at the beginning of the mathematics education courses that the participants were taking at the time of the study. Participation in the study was voluntary. The students were informed that participation would provide insight into their understanding of the concept of function, which would consequently help them in the course they were taking. All of the students enrolled participated in the questionnaire. Here we used convenient sampling. After a preliminary analysis of the questionnaire results, eight PSMTs were invited to participate in the interviews: four Croatian students and four German students. The students were selected based on their results ranging from best to worst. Here we used purposive sampling. The reason for the interview was to gain a deeper insight into the students' concept image.

Educational context

Given our research questions, we describe the background of the student participants. The Croatian students first encountered functions in lower secondary school, where they were presented with tasks involving linear, proportional and anti-proportional assignments, and used graphs and tables to represent those assignments (MZOS, 2006). The linear function and its graph were introduced in Grade 7, and the basic form of the quadratic function and the square root function were introduced in Grade 8 (MZOS, 2006). In

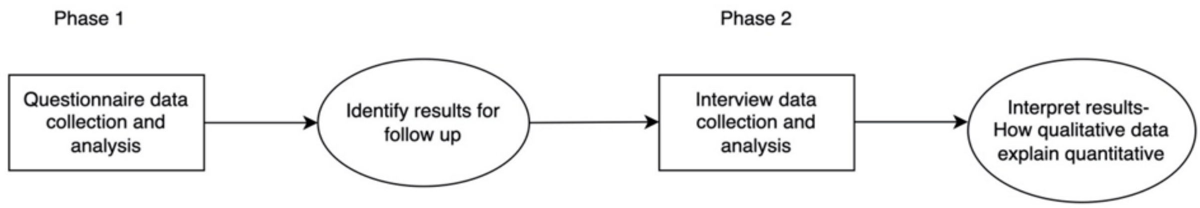


Figure 2. Visual representation of the research design procedure for this study

upper secondary school, students expanded their knowledge of the linear function (Grade 9) and the quadratic function (Grade 10). In Grade 10, they were taught other types of functions such as logarithmic and exponential functions and their properties. In Grade 11, trigonometric functions (sine, cosine, tangent, and cotangent), their graphs and properties were introduced. In the last grade of upper secondary school, students were taught the formal definition of functions and other function properties using the formal mathematical language (MZOS, 2003). In Croatian mathematics textbooks and in school teaching practice, the algebraic approach to functions dominates, that is, functions are represented symbolically with an equation (Gusić & Milin Šipuš, 2019). At university, Croatian students took various mathematics courses (calculus, linear algebra, differential equations, and abstract algebra) where they learnt and applied the formal definition of functions. Upon graduating from university, they can teach mathematics in grades five to twelve.

The German students first encountered functions at primary school, where they were given tasks based on concrete functional phenomena (e.g., verbal description of the relationship amount and cost or patterns in numbers) and mostly presented in the two forms of function representation (as table and diagram) (KMK, 2004). By the end of Grade 6, students should be able to describe and explore patterns and relationships in numbers and quantities (usually using graphs, tables, and diagrams) and make assumptions about relationships between quantities (MSJK, 2004). The foundation of the concept of function is formed at the end of Grade 8 in proportional, anti-proportional and linear relationships, which are represented here by all four types of representation: verbal, graphical, algebraic and tabular. Students should be able to switch between representations and interpret tasks given in the form of graphs and equations, especially in everyday situations. They should also be able to use the properties of the tasks and the rule of three to solve internal and external (real-life) problems (MSJK, 2004). The formal definition of functions is introduced at the beginning of upper secondary school (Grade 9). It is based on the so-called assignment aspect and follows the introduction of functions through numerous examples of everyday general tasks, proportional and anti-proportional tasks and linear tasks (Grade 8 and Grade 9). After that, students learn about linear and quadratic functions. Further expansion of knowledge about functions takes place in Grade 10, where students are exposed to other types of functions such as power, exponential and trigonometric functions (KMK, 2004; MSJK, 2004). At this stage, students have to work out the advantages and disadvantages of the different representations. At university level, the German study participants encountered the function concept in only one undergraduate mathematics course. After graduating from university, they will be able to teach mathematics in grades five to ten.

Data Collection

To obtain as comprehensive an overview as possible of the understanding and identification of functions, as well as the differences of function concept in relation to the curriculum experience, data were collected using two instruments: a questionnaire in the form of open-ended questions and interviews. A mixed methods design was chosen because of its ability to provide an in-depth and complete understanding of the research problem (Cohen et al., 2018). A visual representation (**Figure 2**) was developed to illustrate the research design procedures that were used in this study.

Questionnaire

The questionnaire on function concept (QFC) consisted of four questions. The questions were designed to explore the concept image and concept definition. The PSMTs were given 60 minutes to complete the questionnaire at the beginning of the mathematics education courses they attended. The questions listed in **Figure 3** were given to the students.

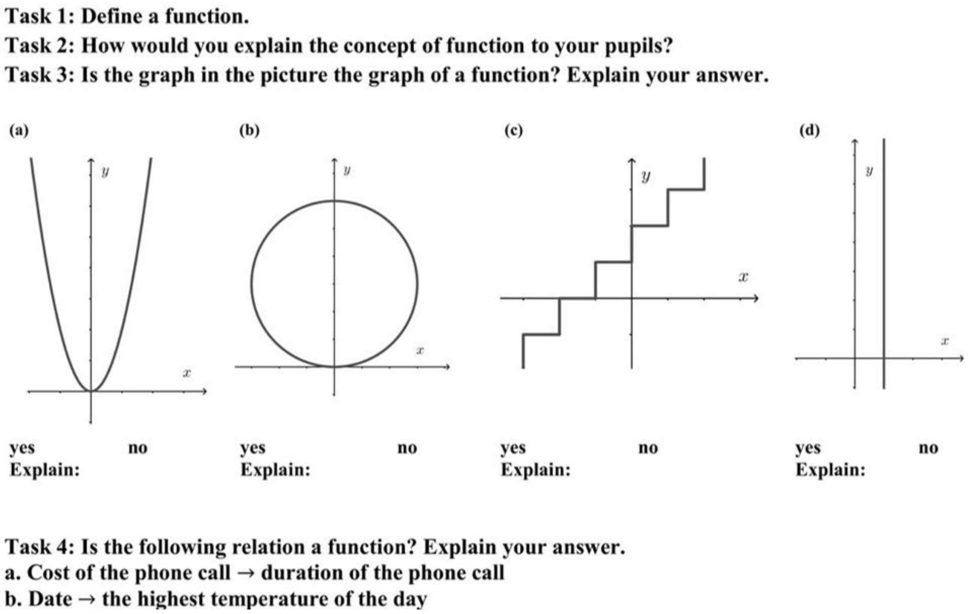


Figure 3. Questionnaire given to students

The tasks for our questionnaire were developed in collaboration with mathematics educators from two curriculum systems; the first author works with German PSMTs (GPSMTs) and, the second and third author with Croatian PSMTs (CPSMTs). The tasks were designed to examine understanding of the concept of function and to reveal misconceptions reported in the literature review. The students did not have access to textbooks or their notes during the questionnaire. Task 1 asks for the conceptual definition of function. Task 2 was intentionally designed to use the word 'explain' instead of 'define' in order to explore students' concept image to some extent, that is, whether the PSMTs would use the formal definition of function or some other representation of function here. The goal of task 3 and task 4 is to explore the coherence of personal and formal concept definitions. Therefore, we intentionally selected several very simple graphs (in particular 3a and 3b) where the PSMTs had to decide whether they represented a graph of a function. Similarly, the relations in task 4 were stated verbally. The PSMTs were asked to explain their answer to avoid pure guessing. Overall, task 1 asked directly for a definition of function, while task 3 and task 4 were focused on its 'indirect' application. Vinner (1991) argued that indirect tasks are necessary to uncover a student's concept image.

Interview

The interviews about the function concept (IFC) were conducted to elucidate how much and in which ways the differences in the teaching of functions in the school curriculums and university programs of the two countries influenced the development of the function concept of the PSMTs. To maintain anonymity, the names of the participants were changed. The complete protocol can be found in the **Appendix A**.

Validity

The QFC and IFC questions were subjected to cross-cultural validity (Cohen et al., 2018). The QFC and IFC questions were first written in German and then translated into Croatian to ensure that students from both countries fully understood the questions asked. We asked two bilingual university mathematics lecturers and mathematics educators to verify that they answer both versions of the QFC in the same way. They also assessed the content of the QFC and IFC questions to confirm that the items tested understanding of the concept of function and to confirm our hypothesis about direct and indirect use of the definition. Here, we used an inter-rater agreement. The rater's comments were also considered in deciding which items to reformulate in the QFC and IFC. All four raters had to agree on the validity of the item to be used in the study. For task 4a, the raters raised the question of whether students know how the duration and cost of the phone call are measured in real life. To determine the appropriateness of task 4a, ten first-year university mathematics students (five from each country) were asked if they knew how the cost of a telephone call was determined in relation to its duration. The multi-method approach with questionnaires and interviews allows

Table 1. PSMTs results

Item number	GPSMTs				CPSMTs			
	Correct	Partially correct	Incorrect	Not given	Correct	Partially correct	Incorrect	Not given
Task 1	13%	48%	32%	7%	71%	26%	0%	3%
Task 3 (a)	55 %	42%	3 %	0 %	73 %	13 %	14%	0 %
Task 3 (b)	50 %	40%	10%	0%	65%	6%	29%	0%
Task 3 (c)	26%	28%	46%	0%	44%	4%	50%	2%
Task 3 (d)	33%	20%	47%	0%	32%	5%	60%	3%
Task 4 (a)	13 %	7%	73%	7%	17%	17%	46%	20%
Task 4 (b)	27 %	3%	63%	7 %	50 %	8%	35 %	7%

the triangulation of data. Denzin (2015) argues that triangulation adds authenticity, trustworthiness, credibility, richness, and depth to any research.

Data Analysis

Classification scaling criteria were developed to evaluate each task from the QFC. This first step of the coding process represented the categorization of the collected data. The following categories were used as classification criteria for task 1, task 3, and task 4: If the answer was given correctly (a correct answer and a correct and clear explanation), it was assigned to the 'correct' category. Answers with an incorrect answer and incorrect explanation were assigned to the 'incorrect' category. If an answer and explanation were not provided, the answer was assigned to the 'not given' category. In addition to these three categories, a fourth category 'partially correct' was created. According to Ron et al. (2010), partially correct means there is the match between a student's construct and the corresponding formal mathematics. All answers that were correct, but the explanation was not mathematically clear enough were placed in this category. Qualitative data analysis of task 1 and task 2 was conducted using qualitative content analysis and the developed content categories (Mayring, 2002). The content categories for this evaluation process in task 1 and task 2 are presented in the results section. All interviews were transcribed and interview data were qualitatively analyzed for each topic: definition of function, examples of function and non-function, school and university experience. The data were compared within the same group of students for similarities and between groups for differences (Strauss & Corbin, 1990). Student statements were used to support results of questionnaire.

RESULTS

In this section, we will present both the quantitative and qualitative results. The results are divided into several subsections, as follows.

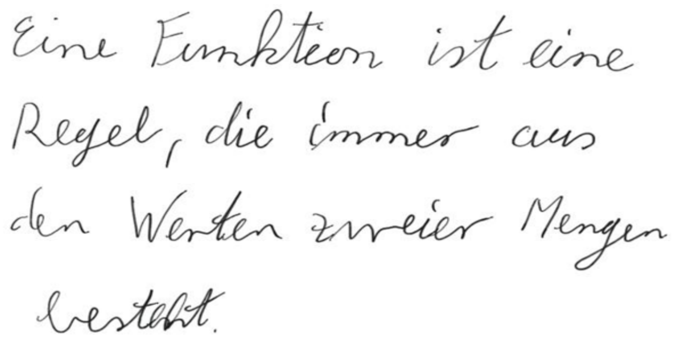
Concept Definition

The results of task 1, which involved the formal definition of function, can be seen in **Table 1**. The majority of CPSMTs provided a clear and correct definition of the function, in comparison, only 13% of GPSMTs were able to provide the correct answer.

The partially correct answers show that the PSMTs disregarded important properties of the function in their personal definition: the property of being total (for all elements in the domain) and univalence (there is a unique image in the codomain). In this category, GPSMTs had about 50% of given definitions. Some of the GPSMTs' answers are, as follows:

> Michael: A function is a correspondence, which relates an element of set A to an element of set B.
>
> Maja: A correspondence between two sets of elements.
>
> Lena: A function is a mapping which assigns an element from set A to exactly one element from set B.

The incorrect answers of the GPSMTs show that students used the concept image when dealing with the function. Their definitions of function included various representations like equation, graph or rules (for example see **Figure 4**).



[English translation: A function is a rule, which always consists of the values of two sets.]

Figure 4. Example of an incorrect answer

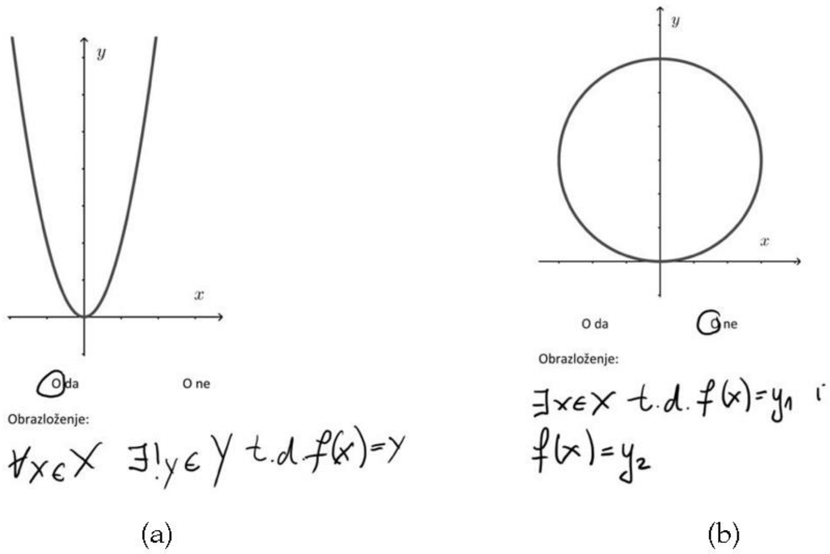
In task 3, which used graphical representations (**Table 1**), the CPMSTs performed better than the GPSMTs, except in task 3d, where the percentage of correct answers was almost the same between the two groups of students. The majority of the CPSMTs in task 3a and task 3b used a formal definition to explain whether the graph represents a function, but additionally used a vertical line test, a part of their concept image, to check.

In the interview, Aneta (CPSMT) stated that she uses a formal definition but also relies on the concept image to examine if the function if given or not:

> "Personally, I always determine the domain and codomain, and then I apply the definition of the function. If the function is represented as a graph, I very often use the vertical test to check if the graph is a function".

In comparison, about half of the GPSMTs answered task 3a and task 3b correctly. Also, many GPSMTs answers fell into in the 'partially correct' category, as they gave no explanation or simply wrote: "A parabola". This result suggests that students stored this graph as a prototype for the quadratic function in the concept image. Another prototype stored in the concept image is a circle as the example of a non-function commonly used in textbooks. The interviewed GPSMTs used precisely this example of a non-function. Lukas (GPSMT) explained: "A circle is a non-function. We mentioned this example on the function course at university".

The PSMTs gave many incorrect responses with non-functions in task 3c and task 3d (**Table 1**). Most of them referred to the function in task 3c as a 'step function' and in task 3d as 'the constant or linear function'. Visually, both graphs resemble a function that the PSMTs would have seen in their school years. For example, the graph in 3d resembles the graph of the linear or constant function. Thus, it appears that the PSMTs did not use the definition of the function to decide whether the given graph represented a function, but relied on their concept image. Most CPSMTs attempted to use the vertical line test inappropriately, which may be the main reason for the 60% incorrect responses in this group. This was also partially confirmed by Aneta's statement above. Several CPSMTs used a symbolic mathematical language to explain whether the given graph is or is not a function (for example see **Figure 5**).



[English translation from left to right: (a) yes; Explanation:$\forall x \in X\ \exists!\, y \in Y$ *such that* $f(x) = y$. (b) no; Explanation: $\exists x \in X$ *such that* $f(x) = y_1$ *and* $f(x) = y_2$.]

Figure 5. Students' responses to task 3a and task 3b

In task 4, which used verbal representations, both groups studied had very low scores for correct answers (**Table 1**). A slightly higher percentage of correct answers was observed for the CPSMTs, who mostly used the formal definition of the function in their explanations. The percentage of incorrect answers is quite high for the GPSMTs. This is surprising because the GPSMT participants were taught the concept of function through numerous examples from daily life in their school and university education. The same was found in the study of Dede and Soybas (2011), who claimed that the lack of adequate theoretical background of the concept of function could lead to problems related to real life situations. In task 4b, many CPSMTs and GPSMTs wrote: "The date and the highest temperature of the day do not depend on each other. The temperature has nothing to do with the date" or "The date and the highest temperature of the day are not related." This indicates that neither PSMT group used the concept definition in that situation, but relied on the concept image. Furthermore, such responses indicate difficulty in recognizing that two variables from daily life can be related, suggesting an incoherence in the concept image.

Overall, both groups of PSMTs have problems when confronted with situations involving graphical and verbal representation of functions/non-functions. However, the comparison of the results, that is, correct and incorrect answers, shows that the GPSMTs are less successful than the CPSMTs. It seems that their personal concept definition is more lacking and presents a greater obstacle in solving the task.

Concept Image

Task 2 provided insight into the PSMTs concept image. When asked how they would explain the concept of function to pupils, students provided a formal definition, a table, an equation, a graph, examples from daily life, and an image (**Figure 6**). About 40% of CPSMTs would use a formal definition to explain the function to their pupils, indicating a strong influence of school curriculum and university education. In comparison, only 19% of GPSMTs opted for the formal definition of function.

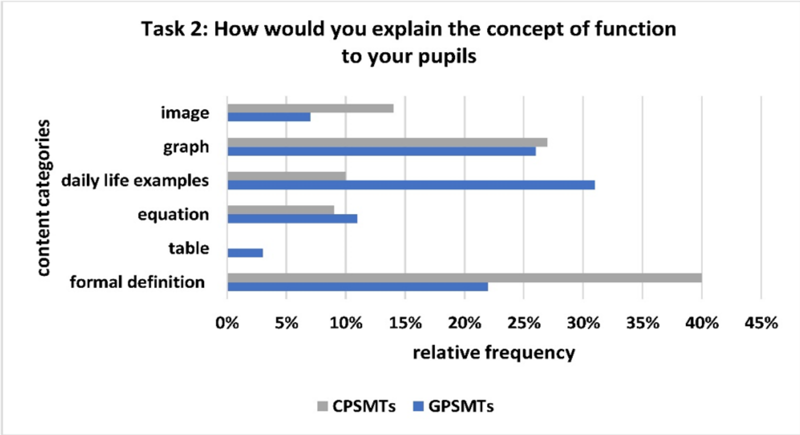


Figure 6. Categories developed for task 2 (blue-incorrect answers of the GPSMTs; grey-incorrect answers of the CPSMTs)

Furthermore, about 34% of the GPSMTs said that they would explain the function using a non-mathematical example, that is, an example from everyday life, compared to only 10% of CPSMTs who used this approach (**Figure 6**). This result indicates the influence of the German school curriculum on the GPSMTs' concept image. This assumption was confirmed in the interviews. All the GPSMTs interviewed immediately and without hesitation gave an example of the function of daily life. As an example, we provide an extract from Anna's (GPSMT) interview:

> Anna: Example of a function? Body height assigned to each person. But, if we look at it another way, it is not a function.
>
> I: OK, and can you give an example of function and non-function in the mathematical sense?
>
> Anna: (thinks for a while) Maybe the one about the circle. Every circle has its circumference associated with it. And a non-function? [thinks] This question is a bit more difficult. Hmm ... Unfortunately, I can´t think of an example. The only thing I can think of now are the examples of a function in the mathematical sense, or the example of a non-function in everyday life.
>
> I: Maybe you can use a representation of the function to find an example that is not a function.
>
> Anna: Maybe as a graph that looks like this. [draws a straight line with the y- axis and explains]
>
> I: Why was it harder for you to find an example of a non-function in the mathematical sense?
>
> Anna: We had many examples at school that were functions of everyday life. And also at university. But not so many of the ones that were not functions, especially not in the mathematical sense.

The interviews conducted with the CPSMTs showed they have mathematical examples of the function in their concept image. However, they showed uncertainty in relation to everyday life examples of the function.

Furthermore, almost the same percentage of both PSMT groups (26% of GPSMTs and 27% of CPSMTs) would use a graph as a way to explain (**Figure 6**). Several PSMTs explicitly mentioned that they would use the graph of the linear function. Linear functions are the first type of functions that students learn in school, so it is not surprising that students remember them as the prototype for all other functions as well as for the definition of the function. In the interviews, the Croatian and German students emphasized that the graph as the representation of function was the one most often used or seen in mathematics classes and textbooks. Anna (GPSMT) also mentioned it in her interview as the first representation of the function. Marija (CPSMT) explained that the typical change from one form of representation to another in the tasks she encountered at school was the change from equation to graph. Below we reproduce an excerpt from Marija's interview:

> I: When I say function, what comes to mind?

Marija (CPSMT): In 8th grade, we were covering the quadratic function and the teacher drew a parabola on the blackboard and that stuck in my mind. When someone says function, I immediately see a graph.

I: Did you also use another way of representing the function?

Marija: In school we also used Venn diagrams to explain what a function is. Functions were represented through the dots, I mean domain and codomain. Then, of course, graphs. And then equations. Yes, now I can remember. In the assignments we almost always had the equation and then we had to draw a graph of the function.

I: And at university?

Marija: Here we also covered the functions with more than one variable. Hmm… Then the modelling with the functions, differential equations and application of differential calculus. We expressed here the functions mostly through the equations.

Concept Image and Concept Definition–A Summary

Participants' concept definition and concept image of the function can be illustrated by **Figure 7**. The concept image of GPSMTs is represented by a blue colored oval and the concept image of CPSMTs is represented by a grey colored oval. According to the data obtained from the questionnaires and supplemented by statements from the interviews, the concept image of GPSMTs is broader and more complete compared to the concept image of CPSMTs. Examples from everyday life dominate in the concept image of the GPSMTs. The graph and the equation prevail in the concept image of CPSMTs. Moreover, the link between the formal and the personal definition of the function is weaker in the GPSMTs (dashed blue line), whereas this link is stronger in the CPSMTs (solid grey line). It seems that many GPSMTs and CPSMTs have an incoherent concept image because the examples they associate with the concept of function contradict the formal system of mathematics. This percentage is a little higher for GPSMTs than CPSMTs. Although not coherent, CPSMTs' concept image shows some consistency with the formal definition.

DISCUSSION AND CONCLUSION

The aim of this study was to investigate and compare the understanding of the function concept of PSMTs from two countries (Germany and Croatia), that is, students' concept definition and concept image. The results indeed showed differences between the two groups of PSMTs; the CPSMTs rely on the formal definition of function more than the GPSMTs, while the GPSMTs have a broader concept image. These results can be related to the educational background of PSMTs, that is, school curriculum and university education. The CPSMTs were introduced to the formal definition of function at the very end of high school and used it on various occasions and in various courses during their university education. Therefore, it seems that the later and more intensive work with the formal definition of function led to the construction of an adequate theoretical background of the concept of function (to some extent), through the formal and symbolic language and aligned the personal definition of function with the formal definition.

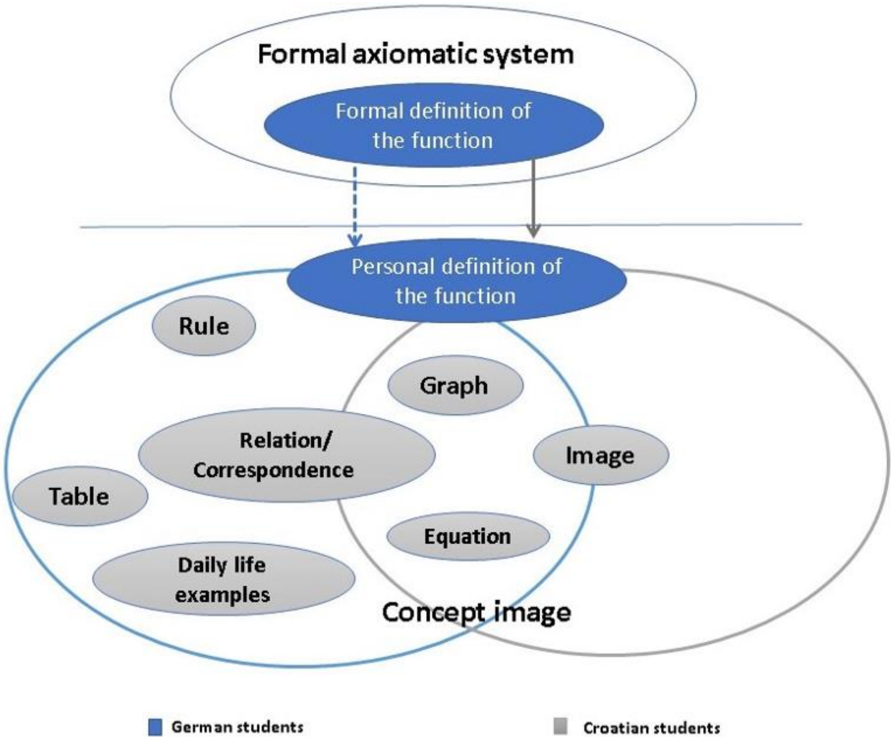


Figure 7. The range of the developed concept image

In contrast, the GPSMTs used various representations of the concept of function in high school and university education and rarely used a formal definition. Thus, it appears that the high school curriculum and university education had an impact on the GPSMTs' personal definition of the concept and the range of concept image. These differences among the PSMTs show what each educational system considers important for future teachers and in which direction they should develop their mathematical knowledge, but also what is considered as relevant mathematics in school. What counts as mathematics, namely mathematical work and mathematical knowledge, has a complex relationship with political, pedagogical, social, and cultural factors (FitzSimons, 2002). Moreover, all strategies and practices in mathematics education are expressions of explicit or implicit values that are shaped by socio-cultural factors (Niss, 1996). Therefore, the mathematics taught in the classroom depends on national curriculum policies, teacher preparation and training, assessment, and daily classroom activities, which in turn affect the development of the concept image and the personal definition of the concept in question.

Within the concept image, both groups of PSMTs experienced difficulties with various ideas as reported in previous studies: disregarding the univalence property (Alajmi & Al-Kandari, 2020; Dede & Soybas, 2011), defining functions as graphs or equations (Dede & Soybas, 2011; Tall & Bakar, 1992;), limiting functions to continuous and familiar situations such as linear and quadratic relationships (Elia & Spyrou, 2006; Hadjidemetriou & Williams, 2002; Leinhardt et al., 1990; Stewart & Reeder, 2017; Stölting, 2008). Here we would like to highlight the issues concerning verbal representations of functions. Both groups of students showed great difficulty with this representation, indicating that this form of representation is not dealt with adequately in either of the education systems. It is certainly something that deserves greater attention, because verbal representation of function links the pure mathematical concept to the external mathematical examples, that is, to the modelling of real life-related examples. Dede and Soybas (2011) emphasized that relating concepts to our daily lives is of great importance for meaningful learning and eliminates the questions in students' minds related to the use of the concept of function.

We do not claim that our results can be generalized or are transferable to other contexts because the sample of students in the study was small and they had specific backgrounds. However, we argue that our results contribute to the existing research on understanding the concept of function. They show that context, that is, the sociocultural factors in which an understanding of a mathematical concept developed, should be taken into account when assessing that understanding.

Implications for Teaching

The learning environment for PSMTs should conceive of mathematics as a human invention that involves connection within mathematics and between mathematics and the real world (Cooney & Wiegel, 2003). This does not mean that mathematical formalism should be excluded, on the contrary, it means that other kinds of experiences should be included and that students should focus on strengthening the connection between concept definition at the formal axiomatic level and concept image. We advocate using tasks that take students out of their comfort zone; tasks that require students not only to recognize but also to discuss whether a specific relation is a function or a non-function.

Relations are a good conceptual infrastructure for developing the function concept because tasks with relations can provide feedback on the evoked concept image for function definition (Hansson, 2006). The focus should also be on examples from everyday life where the domain and codomain are not the set of numbers. In addition, PSMTs should have experience in the process of defining. Edwards and Ward (2004) argue that such experience will be beneficial in future classroom practice. If we want school students to have a better understanding of the function concept, we must first modify and refine this concept with the PSMTs as they will be directly involved in teaching in schools. Therefore, it seems advisable to investigate students' understanding of functions at the beginning of mathematics education courses and further develop students' knowledge based on the results.

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APPENDIX A

The eight IFCs with the PSMTs were conducted in their native language in the form of a face-to-face oral interview or via the online portal Zoom-Platform, where the interviews were audio recorded. The duration of each interview was approximately 25 to 30 minutes and during this time the PSMTs were asked to answer the following questions: "What is a function?", "Can you give an example of a function (a mathematical example and an example from everyday life) and an example of non-function?", "Can you explain or draw what type of representation of a function you encountered most often during your education at school and university?", and "Did you encounter any tasks during your education where you had to determine whether something was a function or not?". When answering the questions, the PSMTs were encouraged to think aloud and feel free to express their initial thoughts to clearly demonstrate their problem-solving ability. The researchers were also allowed to ask several sub-questions to obtain more complete and detailed answers. The transcripts of the interviews were first written in the native language of the students and the researcher, and later translated into English.