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Research Article

The impact of teachers' knowledge on the connection between technology supported exploration and mathematical proof

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ABSTRACT

Technology is recognized for its potential to implement exploration tasks. The ease and speed with which it becomes possible to observe many cases of a situation, allows the development of conjectures and brings conviction about their veracity. Mathematical proof, assumed as the essence of Mathematics, tends to appear to the students as something dispensable. Based on KTMT – *Knowledge for Teaching Mathematics with Technology* model, this study intends to understand the impact of the teachers' knowledge on mathematical proof in a context of technology integration. The study adopts a qualitative and interpretative methodology, based on case study, analyzing the practice of one teacher. The conclusions emphasize the relevance of the teacher's MTK – *Mathematics and Technology Knowledge*, and TLTK – *Teaching and Learning and Technology Knowledge*. The teacher's MTK guides her decisions, leading her to focus on helping students understand the meaning of conjecture and proof, valuing, at the same time, the relevance of algebraic manipulations. However, the teacher's TLTK guides her practice, where the knowledge about the students is determinant. The study provides evidence about the difficulty of articulating proof and technology, but it also clarifies the relevance of this articulation and of how the teacher's KTMT can impact the teacher's decisions.

Keywords: professional knowledge, KTMT, technology, proof

INTRODUCTION

Technology is recognized for its potential for teaching and learning mathematics (Tabach & Trgalová, 2019). In particular, the possibilities of carrying out work of an investigative or exploratory nature are highly valued. It makes it possible for the teachers to offer to the students the opportunity to experiment with different mathematical relationships, reflecting on them while trying to identify regularities and formulate conjectures. However, this possibility challenges the teachers' professional knowledge (Rocha, 2020b). The ease and speed with which it becomes possible to observe many cases of a given situation, brings conviction about the veracity of the formulated conjecture and fosters a feeling that nothing else is needed to be sure of it (Hsieh et al., 2012; Rocha, 2020b). Mathematical proof, assumed as the essence of Mathematics by several authors (Blanton & Stylianou, 2014; Dawkins & Weber, 2017; Rocha, 2019; Schoenfeld, 2009), thus tends to appear to the students as something dispensable (Hanna, 2001).

The potential of technology is also related to the ease of access to different representations (Rocha, 2020a). And, once again, this potentiality challenges the teachers' knowledge. The accessibility and apparent simplicity of the graphical representation turns the algebraic approach into something that can be circumvented and whose need becomes possible to question. The mastery of algebraic calculations, which in an approach without technology was often the only possible option, thus becomes something expendable. It becomes possible to question the interest in learning and teaching certain algebraic manipulations, as well as the level of fluidity and training that should be required from students.

Mathematical proof tends to be related to algebraic approaches (although it does not have to be, as stated by Komatsu, 2010) and the use of technology tends to be related to more intuitive and exploratory approaches based often in graphical representation. As so, not much is known about how to articulate these two approaches. In a previous work by Rocha (2015), it was intended to understand how the teachers conceive proof and an algebraic approach in a context of technology integration, and how they try to turn the algebraic approach relevant to the students. In the present study, the goal is to understand the impact of the teachers' knowledge on mathematical proof in a context of technology integration. However, the focus is not exactly on the proof itself, but more on the understanding about what a proof is (what characterizes it and how it differs from a conjecture). The study adopts the KTMT (Knowledge for Teaching Mathematics with Technology) model (Rocha, 2020b), giving a special attention to the MTK (Mathematics and Technology Knowledge) and to the TLTK (Teaching and Learning and Technology Knowledge) – two of the main knowledge domains in the KTMT model, as discussed in the next section. Based on this conceptualization and considering the use of exploration tasks[1] in the study of functions in 10th grade, this study intends to answer the following research questions:

- What is the impact of the teachers' TLTK in mathematical proof while implementing explorations in a context of technology integration?
- How does the teachers' MTK influences the decisions related to mathematical proof while implementing explorations in a context of technology integration?

A better understanding of the teachers' professional knowledge will offer a deeper understanding about how mathematical proof and conjectures are addressed in exploration tasks with the use of technology. And knowing how TLTK and MTK impact the teachers' practice will be very important to promote the teachers' professional development.

Mathematical proof

The literature about mathematical proof has devoted attention to several topics, some of them focusing on the students and some others focusing on the teachers. In what concerns teachers, the research has focused on ways of addressing proof in the classroom and on the teachers' knowledge and professional development (Stylianides et al., 2016, 2017). Nevertheless, and besides all the interest in different topics related to proof and its teaching and learning, not much attention has been given to proof in a context of technology integration.

The understanding about what a mathematical proof is, has changed over time (Smith, 2006), and is not consensual even among mathematicians (Miyakawa et al., 2017; Steele & Rogers, 2012). Steele and Rogers (2012, p. 161) assume proof as "a mathematical argument that is general to a class of mathematical ideas and establishes the truth of a mathematical statement based on mathematical facts that are accepted or that have been previously proven". Bleiler-Baxter and Pair (2017, p. 16), inspired by De Villiers's (1990) work, define proof as "logical deduction that is used to verify, explain, systematize, discover, and communicate mathematics". In the classroom context, Stylianides and Ball (2008) refer to it as a mathematical argument that uses mathematical knowledge considered valid by the students and that does not require additional justifications, it adopts reasoning considered valid and already known by the students (or whose understanding is within their reach), and which is adequately communicated in ways already familiar to the students (or whose understanding is within their reach). The focus on the classroom, present in this understanding of proof and, specifically, the consideration given to the students' level, make this understanding of proof suitable for this study. As so, this will be the understanding assumed.

The difficulty in getting students to understand the need for and importance of proof in Mathematics is, according to De Villiers (1999), well known to all secondary school teachers. This difficulty is accentuated when technology is involved because, according to Hsieh et al. (2012), the dynamic character usually offered by it

[1] Here, authors assumed as an exploration task, a task where the students analyze different situations, trying to infer some regularity, to develop a conjecture.

allows the carrying out of work of an experimental nature, which enhances the discovery of properties and the formulation of conjectures. Students can easily experiment and analyze various cases, reflecting on important mathematical ideas and, consequently, reaching a higher level of understanding (Goos & Bennison, 2008). Thus, they acquire the possibility to formulate their own questions and to continue formulating hypotheses and testing them, trying to frame the results in the theory they are trying to formulate (Rocha, 2015).

The way in which the analysis of different cases is made possible, ends up giving students a feeling of confidence regarding the veracity of the conclusions they establish with the support of technology, which is often enhanced by the way students got used to seeing Mathematics validated, i.e., externally, either by the teacher, the textbook or even the parents (Tall et al., 2012). The need to prove the formulated conjecture may thus not be felt. But if inferring a conclusion from reflection on some particular cases is an important activity, it is undoubtedly distinct from proving (Cabassut et al., 2012). Emphasizing to the students the need for and importance of proof will then imply the search for its function.

De Villiers (2012) considers that, traditionally, the justification or convincing about the validity of a conjecture is seen as the main function of proof, and Knuth (2002) considers that this is really the only role that most teachers recognize to it. In recent decades, this narrow view of the role of proof has been criticized by authors such as Reid (2011), who understand that it has also assumed other important roles for mathematicians and that it can also assume a role of great didactic value in the classroom.

For Mejía-Ramos (2005), the search for a deeper understanding is what truly moves mathematicians and what leads them to reject the "alleged" proofs carried out by computational means. A point of view also shared by Bleiler-Baxter and Pair (2017). And this, as highlighted by Hanna (2014), despite the fact of understanding being something remaining relatively undefined. This suggests a role of proof as a means and not so much as an end in itself, encompassing both validation and understanding. In the current reality, in which systems with symbolic algebraic calculus and dynamic geometry programs are easily accessible, it is frequent to obtain a validation of the conjecture with a considerable degree of confidence without a proof (De Villiers, 2012). As so, it becomes difficult to justify the need for a proof exclusively with the need for validation.

Technologies can convince us of the veracity of the conjecture, but they do not offer us the justification for that veracity (De Villiers, 2012). And this does not seem to be a question exclusive for mathematicians. Indeed, a study conducted by Healy and Hoyles (2000), in the context of algebra teaching, suggests that students prefer arguments that simultaneously convince and justify the relationship in question. A conclusion suggesting that explanation is something important for students and that it can even be a worthy resource for greater use and exploration in the teaching of Mathematics. Interestingly, the situation seems to be interpreted a little differently by some teachers. Indeed, as mentioned by Biza et al. (2010), while all teachers recognize the verifying role of proof, the same does not happen in relation to its role in terms of comprehension. Actually, as the authors refer, some teachers tend to check the validity of a mathematical relationship based on examples, even when they have just proved it. Besides that, teachers consider that arguments based on concrete cases or on visual representations have greatest potential to convince.

But there are other roles that are also assigned to proof. Bleiler-Baxter and Pair (2017), and several other authors, refer to proof as a discovery process (a function of proof introduced by De Villiers, 1990, 2020). According to all these authors, there are numerous examples in the history of Mathematics of new results that were discovered or invented by purely deductive processes; in fact, it is completely unlikely that some results (such as, for example, non-Euclidean geometries) could ever have been found by mere intuition. The role of proof as a systematization process is also addressed, considering that it reveals the underlying logical relationships between statements in a way that pure intuition would not be able to accomplish. In turn, Davis and Hersh (1983) see proof as an intellectual challenge, considering that it fulfills a gratifying and self-fulfilling function. Proof is therefore a testing ground for intellectual energy and mathematical ingenuity.

Thus, the literature highlights the need to better understand the articulation between explorations made with technology and mathematical proof, suggesting difficulties on the part of teachers in this articulation. It also points to different functions of proof, identifying different potentialities, but also showing the existence of different valuations by teachers. And this are issues somehow addressed in this study and closely related to the teachers' professional knowledge.

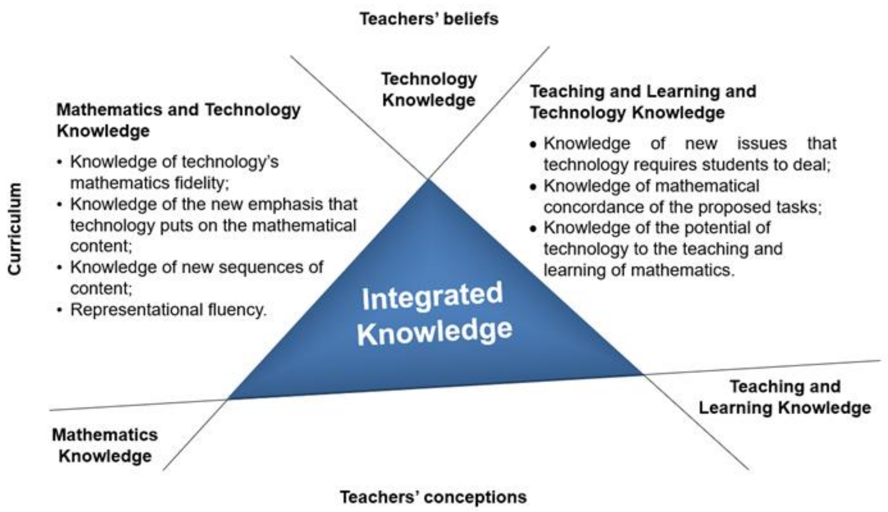

Figure 1. KTMT model by Rocha (2020b)

Knowledge for Teaching Mathematics with Technology – the KTMT model

The main goal behind the conception of the KTMT model is the articulation of the research about the teachers' technology integration and the research about the teachers' professional knowledge. The model recognizes the contribution of the work of authors such as Shulman (1986), and Mishra and Koehler (2006) on the definition of the knowledge domains considered and assumes three types of knowledge domains: base knowledge, inter-domains knowledge, integrated knowledge.

The base knowledge domains are four: Mathematics, Teaching and Learning, Technology, and Curriculum and Context. Curriculum and Context is assumed as a transversal domain, influent on all the other domains. This is a domain that includes all the influences over the teachers' options, being these personal influences (such as the teachers' beliefs) or external influences (such as the school context).

Inter-domain knowledge is a type of knowledge central in this model and the main characteristic of it, as well as the main difference from other knowledge models. This type of knowledge is a new knowledge developed from more than one base knowledge and integrating in its characterization results from the research on technology integration. The KTMT model considers two inter-domain knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and Technology Knowledge (TLTK) (**Figure 1**). MTK focuses on how technology influences mathematics, enhancing or constraining certain aspects, and TLTK focuses on how technology affects the teaching and learning process, enhancing or constraining certain approaches.

Integrated Knowledge (IK) is the last type of knowledge in the KTMT model, developed from the articulation between all the knowledge domains. As the previous mentioned domains of knowledge, this is a new knowledge. It develops from the knowledge held by the teachers in the base domains and in the inter-domains, however, this development does not prevent the continuous development of the knowledge in all the domains. This is an on-going process. The knowledge in all the domains continuous to evolve, generating new knowledge and contributing to the professional development of the teacher.

Integrating knowledge from different domains, such as Mathematics, Teaching and Learning and Technology is assumed to be central in the KTMT model. An option also presents in other models, such as the TPACK from Mishra and Koehler (2006). However, the way how this integration is conceived is different. And this is a very important characteristic of KTMT and the main difference of this model in comparison to others. MTK and TLTK are not conceived as knowledge resulting from an intersection of knowledge in the base domains. They are new knowledge. A new knowledge resulting from an articulation between two of the base knowledge domains. And this is a dynamic knowledge, a knowledge that continues to be developed, as knowledge in two of the base domains continues to interact and to generate some new knowledge.

The research conducted so far on technology integration has offered some very relevant results. KTMT intends to integrate these results on the model. For instance, the research on technology integration

Table 1. Analysis criteria

MTK		TLTK	
Knowledge of the Mathematics and of the technology impact on it	Knowledge of how technology enables the discovery of mathematical relationships and regularities Knowledge of how technology, by allowing the observation of many cases, can affect the relevance of proof, reducing or even eliminating it	Knowledge of the teaching and learning and of the technology impact on them	Knowledge of the characteristics and potential of exploratory tasks in the context of technology integration Knowledge of students' difficulties in the context of technology integration

documents students' difficulties, and the KTMT model includes the teachers' awareness of the difficulties faced by the students when using technology as part of the teachers TLTK. There are also studies addressing how technology can impact the mathematics content being addressed, and the model includes knowledge about the new emphasis technology can put on the mathematical content as part of MTK.

TLTK and MTK are the inter-domain knowledge, and they have a central role in the model. As so, they will have a central role in this study.

Methodology

The investigation presented here adopted a qualitative and interpretive approach, based on a case study (Yin, 2017), and focused on the teacher called Teresa. Data collection involved interviews, observing a 10th grade class while studying functions and collecting documents. Semi-structured interviews were carried out before and after each class observed, with the intention of knowing what the teacher had prepared and the reasons for these options (pre-class interviews) and her reflections of the way the class took place (post-class interviews). 14 lessons, where the teacher was planning to use technology, were observed while the students were studying functions of several types (linear, quadratic, absolute value, defined by branches). Both the interviews and the lessons were audio-recorded. A logbook of the observed lessons was also prepared and documents such as worksheets and other materials made available by the teacher to the students were collected. Data analysis was essentially descriptive and interpretive.

Data analysis was based on the criteria presented in **Table 1**. These criteria were developed from the KTMT model attending to the characteristics of the present study, namely the focus on proof. These criteria were then used to interpret the options assumed by the teacher. As a first step, the teacher practice in the classroom was divided in parts (such as launching the task, providing information, supporting the students) and then each part was analyzed intending to identify evidence of the defined criteria. So, each of the parts identified was analyzed using successively the criteria presented in **Table 1**, to identify evidence of the teacher's MTK and TLTK.

The participant in this study was a teacher with over 30 years of professional experience. She had a long experience of using graphing calculators with students (the technology used in the study and owned by each of the students) and a deep knowledge of the machine's operation. During this study she taught the topic Functions in Mathematics to a 10th grade class at a school in Portugal.

The teacher was aware of the students' limited experience as well as the difficulties faced when requested to produce a mathematical proof. As so, she had been proposing to her students some tasks where she asked for the development of mathematical arguments and, sometimes, for a proof. This was done during the study of Geometry. Since the beginning of the school year (about two months before the implementation of this study) the students were also becoming familiar with exploration tasks and the development of conjectures. In these tasks the students were expected to explore several examples and identify regularities, being the formulation of the regularity identified the conjecture.

RESULTS

In this section we present one of the tasks (see **Appendix**) proposed by the teacher and where, in addition to formulating a conjecture regarding a mathematical situation, students are asked to prove their conjecture (T-teacher, S-Student, R-Researcher).

Teresa starts the lesson informing the students that they are going to carry out an exploration task and that this work will be carried out in pairs. She emphasizes this last aspect, stressing the importance of the collaborative work. This approach gives evidence of the teacher's awareness of the characteristics of this type of work, also suggesting knowledge about the need to share with the students some of these characteristics (TLTK).

She then gives some information regarding the operation of the calculator, focusing on what she considers that the students will need during the task. The technical knowledge of the technology is shared in this way with the students (TK). Then, she shares her expectations, speaking about which questions she considers will be easy, which ones could be more difficult and how far she wants everyone to go. An action showing knowledge about this type of tasks, but also about the students and the easy way how they can lose notion of time (TLTK):

> T - The aim of each pair is to do everything up to question 6. Up to question 5 I think it's easy. You must do well, as quickly as you can. Question 6 will not be so easy, (…) here it is expected that you prove. I think the proof is not very difficult and therefore I have some hope that many of you will be able to do the proof. The "Going further", which comes in questions 7 and 8, I also hope that some of you manage to do it. If some of you manage to do these questions, it's very good because I don't hope that you have time to do it here in class, but I hope that you do it at home, afterwards. So, the goal is for everyone to do everything up to question 6, including the proof, for some the goal is to do also question 7 and then, who knows… (lesson)

Before encouraging students to start working, the teacher also addresses the issue of proof and its relevance in Mathematics, briefly discussing central ideas in Mathematics (MK), but also connecting them with the impact of using technology (MTK). In this approach, Teresa emphasizes to the students the need to some kind of confirmation before assuming the veracity of a conjecture (TLTK):

> T - The sixth question (…) is a proof and I would like to talk a little bit about it. (…) In Mathematics we often experiment. We've already done that here with functions. We have studied families of functions and then or I give you some information, saying that the conjecture you formulated is true in all cases, and you believe me, you can also consult the textbook and etc., or we prove the result is always true. We do what mathematicians always do. In Mathematics, proof is the essence of the discipline, so we cannot forget about it. (lesson)

From this moment on, the entire lesson takes place centered in the students' work, with the teacher circulating among the groups and responding to their requests.

The first conjecture of one group of students was based on two examples and states that the line passing through two points of the parabola defined by $y = x^2$ will cross the y-axis at the symmetric of the product of the abscissas of the two points. Being two observations a very small number, Teresa feels the need to draw the students' attention to that, trying to call their attention to the risk of establishing conclusions based in the small number of examples that were considered in their formulation. But the students do not seem very sensitive to her comments and only the doubt about the veracity of the conjecture seems to lead them to consider analyzing a few more cases:

> T - Are you formulating a conjecture based on just two examples?
>
> S - Oh, but we've already seen it.
>
> T - And what did you notice?

S - It corresponds to multiplication, but it has to be less this times this. (...) It has to be –(5 × 3).

T - Okay, great. It's your guess.

S - (...) But that's -15. It's wrong. That's why in the next question they ask for an answer if the points are in the same side of the axis. Isn't it?

T - I don't know. (...) You only experimented with two examples. You are taking conclusions based only in two examples... you can see more examples, if you have doubts. That way you can check if you are getting it right or not.

S - How many pairs should we do?

T - In an investigation there is no limit. Do several, until you can reach a conclusion... two is very little to do. I think, don't you? (lesson)

Seeing the quantity of cases analyzed to develop the conjectures, the teacher tries to let the students think about the confidence they can have in the result formulated. But seeing they are not sensitive to that, and knowing the importance of letting them explore, she chooses to instill the doubt in their mind (TLTK).

Not all the students react this way. Some consider that the more examples they do, the better. But even so, they seem to feel some discomfort for not being given a specific number. And once again, the knowledge of the teacher guides her action (TLTK) and makes her avoid giving a direct answer and leave the decisions to the students:

S - How many [examples] should we do?

T - That’s up to you.

S - As many as we wish. The more the better... (lesson)

But in some cases, in addition to the number of examples considered, the conjecture seems to be formulated in a somewhat thoughtless way, leading Teresa to question the students so that they feel the need to better ponder the conclusion they reached. Once again, the teacher poses questions, instead of giving answers, leaving the exploratory work to the students (TLTK):

S - I have already concluded something. The ordinate at the origin is always $x_1 \times x_2$ and then the slope of the segment is the difference between one and the other.

T - $x_1 \times x_2$? So how much is it $3 \times (-5)$?

S - No.

T - Tell me, how much is it?

S - -15.

T - -15, and there it is?

S - 15.

T - $3 \times (-4)$?

S - It's -12. So... okay, it's the other way around, it’s the reverse.

T - The reverse?

S - Yes.

T- Is it the reverse?

S - Yes. Is it the module?... It could be less. The ordinate at the origin is less or…

T – So, think about it… but write the conclusions. (lesson)

The proof was the final phase of the work carried out in the lesson by the students, as predicted by Teresa, once none of them managed to go beyond this in the available time.

This was a phase of the work in which difficulties arose, something that Teresa already anticipated (based on her TLTK) and which, as it happened, she intended to address individually, supporting the students as the problems arose:

T - The proof, even in the simplest case, is still not simple for these 10th grade kids. I will have to give some tips on the spot and there will be some that do it and there will be others that take a long time. (pre-lesson interview)

While addressing the question related to proof, however, other issues arise. The first one concerns the meaning of the term conjecture, with different students questioning its meaning, even after having already elaborated their conjecture:

S1- Teacher, what is the conjecture?

T - The conjecture is exactly that. That's what I think will be true. Afterwards, I must prove it. I think it's true, but I need to prove it really is. While studying Geometry we did that. Here, in the cases you have seen, it is true (referring to the examples considered by the students) and this allows me to conjecture, it allows me to think that it will always be true. It's only when I prove that I'm sure it's always like that. It is true in all the cases.

(…)

T - What is the conjecture? What do you want to conjecture?

S2- But what are we supposed to say by conjecture? (lesson)

But understanding the meaning of the term proof seems to be even more complex. Indeed, some students seem not to feel the need for generic analytical work, when the cases they analyzed leave them convinced of the truth of their conjecture:

S - And here in question 6, if we have already shown the calculations here (points to the examples recorded above)… Can I say that this proves the validity of our conjecture?

T – Does it?

S - No? (lesson)

In fact, instead of trying to prove their conjecture, what many students did was to perform analytically the calculations for the slope and the ordinate at the origin of the cases they had considered graphically. Even so, they have doubts if this is really what is intended:

S - We are not understanding question 6.

T - It’s the proof.

S - Do we do the math? Should we put the calculations?

T - Right. But you did it for these three cases. Now, for a proof… (the student interrupts her)

S - Ah! We must do more!

> T - A proof… I mean, to be proved I have to do it for how many cases?
>
> S - For many.
>
> T - How many? How many?
>
> S - Infinite.
>
> T - Infinite. (interrupts to ask for silence to the class and then helps the students to find a way of representing a point in a generic form)
>
> S - It's complicated.
>
> T - It's complicated… but we don't give up of something just because it's complicated. (…) The proof must be analytical, and that it's not possible in the calculator… You can try to see many cases, but you cannot see infinite cases. (lesson)

The teacher is expecting the students' difficulties (TLTK), but she is also prepared for the students view of proof as something unnecessary (MTK). Teresa considers this is a natural approach for the students, as it follows on from what they have been doing:

> T - I saw, I don't know how many students… now I'm going to see what they wrote, but there were some students that in the proof… what did they do? They move to an analytical approach. They approach the same examples, but now using analytical calculations instead of using the calculator. (…) And this basically corresponds to what we have done in other situations. We don't call it a proof, of course, but it corresponds to work we have done. I have been concerned about working in the calculator and working analytically and therefore I think they have made a transposition of these situations that we have been doing… here for this. (post lesson interview)

The articulation between the graphic and the analytic is, therefore, something that Teresa says she pays attention to and that she addresses in the challenges she poses to the students at the end of this task and which she intends to explore in another lesson. Indeed, these last questions come precisely to emphasize the relevance of this choice between the graphic and the analytic approaches. The teacher considers that students generally prefer the graph approach over the analytical, thinking that the latter is just calculation without much usefulness (TLTK). In this case, however, the analytic approach offers the simplest and quickest approach to the question, although not necessarily an easy one (MTK). And the teacher wants her students to be aware of that:

> R - In "Going further" the parabola becomes another. Do you think it's easy to experiment some cases with the calculator and discover the relationship?
>
> T - No, I don't think so.
>
> R - It's just that I didn't make it. I found it, but I found it analytically. It's also true that I got tired. I gave up and decided to do it analytically.
>
> T - Exactly. But the intention is also that. It's for them to realize that there are things where we don't need to go into calculus, but there are others where calculus is useful. And this calculation is still difficult for them, isn't it? But I prefer to work the calculus like this, so that they realize that there is some advantage in doing some calculus… (pre-lesson interview)

The notion that, in order to prove, it is necessary to consider all the cases and not just a few (MTK) is something that she believes needs to be worked on over time (TLTK). In this task her main goal is to make the students aware of the relevance of proof even when the technology already convinced them about the veracity of their conjecture (MTK), starting from the students' conceptions that she is anticipating (TLTK):

> T - I expected them to have difficulties in the proof. (…) The idea is exactly to go on with this discussion with them… then I… as I gave them until Wednesday to finish all the questions in the task, so it will probably be in the Wednesday lesson, I will give back to them what they wrote, and we will go back to the discussion about the difference between trying one, two, three cases or doing… (…) And I will discuss with them mainly this question: what does it mean to prove. The task asks them to include the examples they've already done, but it also asks them to prove. And that means consider all cases and, in this case, they were infinite. (post lesson interview)

In this sense, she even expresses her intention not to close the issue yet. Discussing with the students the proof in the simplest case and leaving the challenges open, to be presented later to the class by some of the students who can solve them. And the teacher makes considerations about the right moment to do it (TLTK), referring to a moment when the calculations needed to prove are being a focus of the lessons (MTK):

> T - I'll do the proof in this case, just for $f(x)=x^2$, and I will leave the challenges of "Going further" still open. As they manage to address the challenges, they can write what they did and give it to me. (…) Doing it requires some algebraic manipulation of expressions and they have never worked on it because in the previous school years we don't do this kind of work up to this level. As we are now starting to study the polynomials… The idea is to make them aware of the relevance of these algebraic manipulations, instead of addressing it disconnected from any relevance. So, later, I intend to go back to this, when some of them have already done it. I'll ask one of them to make a presentation to the class, when we are working on calculations with polynomials. (post lesson interview)

After trying to make students realize that proving requires that all cases are considered and not just a few, Teresa chooses to help students to consider generic points that allow them to effectively prove what is intended. She supports the students work in what she knows they already can do (TLTK) and tries to make them going forward, supporting them in finding a suitable representation and connecting it with their mathematical knowledge and what they experienced with technology (MTK), inspiring them to move from the particular cases to the general one:

> T - So in question 6 what I'm asking is this: for these points this is true, so now following this reasoning, if the points are not these… You have two points, then what if it is a point 1, for example, of coordinates (x_1, y_1) and a point 2 of coordinates (x_2, y_2). Now this y_1 and this y_2 are not just any ones. Why? These points also belong to the parabola. And so, what is it, what is y_1? And y_2? (helps the student to get to the answer) So this point is $(x_1, {x_1}^2)$ and this point is $(x_2, {x_2}^2)$. (…) Will you now be able to prove? Now prove… you must use what you know. You know how to calculate the slope of a straight line passing by two points, right? So, let's try to do it.
>
> S - But here, up here we had already shown this.
>
> T - You showed, but that's just for one specific case. If you show for this case… you have to do exactly the same reasoning, but the calculations are a little more complex, you have to do it slowly and carefully… If you do the same reasoning but for any point, you don't show it for one single case, you show it for how many cases?
>
> S - To infinite. (…)
>
> T - So if you can do exactly the same reasoning but for this general case… (lesson)

It is possible to see that during all the task, the teacher is balancing her approach guided by her TLTK and her MTK. In one hand the teacher is supporting her options in what she knows about this type of tasks and about the students' approaches and difficulties and, in the other hand, she is being guided by the mathematical knowledge she wants to promote, keeping in mind the potential of the technology. This suggests the teacher is guiding her practice by her IK.

CONCLUSION

The main goal of this study is to understand the impact of the teachers' knowledge on mathematical proof in a context of technology integration, giving a special attention to the impact of the teacher's MTK and TLTK.

The Teachers' MTK Influence in the Decisions related to Mathematical Proof While Implementing Explorations in a Context of Technology Integration

The teacher's MTK guides her decisions, leading her to focus on helping students understand what a conjecture is (where the need to ensure its validity deserves emphasis, as addressed by De Villiers, 1999), and what a proof is. The main focus seems to be on this understanding rather than on the proof itself. Still, there is the intention to help students adopt a more formal language (with all the challenges included, Aristidou, 2020), important for the realization of a proof (where the teacher tries to help the students to considerer a general point and not a specific one). This domain of knowledge is also responsible for her intention to help students understand the importance of algebraic manipulations, making them feel that it is not just calculations and procedures that they have to learn, but that there is a use for them (present in the way how the relevance of proof is presented to the students, but also in the challenges at the end of the task and left to a later moment).

The way how proof is integrated in the task, after a stage of exploration and conjecture formulation, and with a focus on ensuring the validity of the conjectures, ascribes to the proof the role of verification. Roles such as the one of understanding are not considered by the teacher in exploration tasks. However, this option can be more a result of the type of task than of the teacher's MTK. The evidence available does not allow us to conclude that the teacher is not aware of the different roles of proof addressed in the literature (De Villiers, 2012) or even that she does not value them (Knuth, 2002).

The Impact of the Teachers' TLTK in Mathematical Proof While Implementing Explorations in a Context of Technology Integration

Although there is clearly a focus on Mathematics and a set of learnings focused on Mathematics, the teacher's choices seem essentially guided by her TLTK. And this is because it is the teacher's knowledge of the students and their difficulties that seems to guide all the decisions. It is the teacher's knowledge of the type of task (as suggested by Rocha, 2020b) and the way in which the students approach them (often advancing and establishing conclusions based on a very small number of observations) that leads her to reinforce the importance of validating the conjectures, in line with the work of Hsieh et al. (2012) (trying to make the students understand the relevance of thinking carefully, based on a set of cases, before formulating a conjecture; and transmitting the idea that a conjecture is something that seems to be true, but requiring a deeper analysis -the proof- before it is possible to be sure it is always true). And this is a decision that is based on the knowledge of the students, but also on what is the essence of Mathematics, as assumed by Blanton and Stylianou (2014), Dawkins and Weber (2017), Rocha (2019) and Schoenfeld (2009) (the teacher is aware about how the students can be convinced of the validity of a result based on the observation of some cases; but she also knows the relevance of proof in Mathematics). Thus, although the teacher's TLTK is the starting point that guides her practice, an IK is actually present. It is also the knowledge that the teacher has of the students that leads her to define the understanding of the need for proof as fundamental (when designing the task, the teacher decides to go forward and does not accept to finish the work with the students development of the conjecture) and to recognize that this is still a complex process for the students and that it must be progressively developed (realizing that the students need help to write a general point, and understanding the difference between conjecture and proof as a first step and the proof as a challenge for most of the students). But the importance of insisting on this aspect, an issue addressed by Cabassut et al. (2012), comes from her MTK and so, once again, it is possible to identify an IK. The knowledge about the students' preference for graphical over analytical approaches is also part of the teacher's TLTK (she is expecting that the students do not feel the need to prove, convinced by what they observed with the technology). But the teacher's MTK allows her to be aware of the importance of both approaches and, in conjunction with her TLTK (and therefore IK) leads her to deliberately look for opportunities to confront students with situations where both approaches prove useful.

Final Comments

The knowledge about the relevance of proof in Mathematics, together with the need to understand what a conjecture is and the difference from a proof; as well as the knowledge about the students and their difficulties, are part of the teacher's MTK and TLTK and guide the teacher's action. The integration made by the teacher between TLTK and MTK (i.e., IK) seems to be of great importance, as it allows the characteristics of an exploratory work not to be abandoned, having the students effectively experimenting and seeking for regularities (TLTK), but, at the same time, it allows to approach the essential characteristics of the Mathematics, namely the need to guarantee the veracity of the conjectures formulated in all cases and not only in those observed (MTK). It seems, therefore, that it is the articulation between the two domains of knowledge at IK that allows for a balance that enhances student learning.

The study provides evidence about the difficulty of articulating proof and technology, in line with the difficulties addressed in the literature and related to mathematical proof (De Villiers, 1999; Hsieh et al., 2012), but it also offers evidence of the relevance of this articulation and of how the teacher's professional knowledge can impact the teacher's options. It would be interesting to deepen the articulation between proof and technology, paying attention to other types of tasks (beyond exploration tasks) and looking for understanding about how different roles of proof (beyond verification) are/can be addressed.

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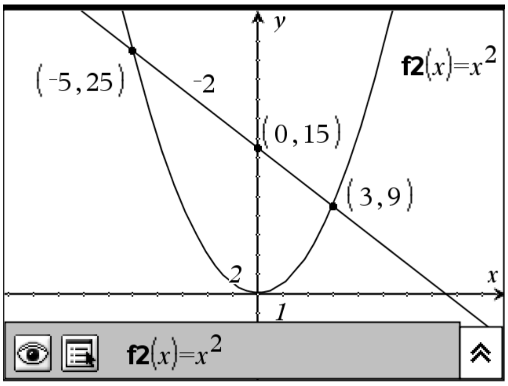
APPENDIX

On the Parabola's Axis



Consider the quadratic function defined by $f(x) = x^2$.

1. Represent it graphically in the window: xϵ[-10, 10] and yϵ[-8, 30].
2. Choose two points on the parabola, one on each side of the vertical axis. For example, points x_1 and x_2 of abscissas 3 and –5.

Draw the line joining these two points.
Record the ordinate at the origin and the slope of this line.

3. Repeat the process for other pairs of points with abscissas of your choice and fill in this table:

Abscissa of x_1	3
Abscissa of x_2	–5
Slope of the segment	
Ordered at origin	

4. Make a conjecture about the relationship between the slope of the segment and the abscissas of x_1 and x_2.
5. Make a conjecture about the relationship between the ordinate at the origin and the abscissas of x_1 and x_2.
   Will the conjectures be valid if the two points are on the same side of the axis? Confirm.
6. Demonstrate the validity of your conjectures.

Going further

7. What would happen with the function $f(x) = 2x^2 + 5x + 6$?

Going even further

8. And in the general case of the function $f(x) = ax^2 + bx + c$?

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