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Research Article

The development of mathematical knowledge of prospective primary teachers in a lesson study

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ABSTRACT

This research aims to understand how lesson study may promote the development of mathematical knowledge in prospective primary teachers, more specifically, what aspects of mathematical knowledge they mobilize or develop and how. The research follows a qualitative approach and the interpretive paradigm and was carried out in a Portuguese institution where lesson study was undertaken in the final year of the initial teacher education of prospective teachers. The data was collected through video recording of the sessions and recording in a logbook, semi-structured interviews and document collection. Four prospective teachers, two teacher educators, and the researcher participated in the lesson study. The research lesson focused on numbers and operations. The results suggest that lesson study is a formative process with potential to develop the mathematical knowledge of prospective primary teachers, by providing an integrated and collaborative setting for this development and enriching experiences through its opportunity to plan-observe-reflect that enables a greater connection between theory and practice. The prospective teachers mobilized and developed mathematical knowledge about the topic, the structure of mathematics, and mathematical practice, through their involvement in lesson study activities, namely: (1) solving and discussing adaptations of tasks and anticipating students' strategies and their possible difficulties; (2) discussing among prospective teachers and teacher educators about detailed aspects of the planning of the research lesson and reflecting on the lesson events and students' learning; and (3) sharing ideas and discussions with the teacher educators guiding the process.

Keywords: lesson study, mathematical knowledge, initial teacher education

INTRODUCTION

In initial teacher education, prospective teachers should develop and deepen different aspects of teacher knowledge, particularly of mathematical knowledge. This knowledge, especially in prospective primary teachers, has shown weaknesses (Strutchens et al., 2016). In addition, initial teacher education faces other challenges and problems that have been pointed out throughout the years and which should be minimized (Branco, 2013). Therefore, it is important to look for ways to minimize these difficulties in initial teacher education.

This study is the development and deepening of a paper first presented at CERME 2023.

One promising possibility is provided by lesson study (Ponte, 2017). Lesson study is a formative process in which a group of teachers or prospective teachers plan a lesson in detail, which is taught by a member of the group and observed by the others, and on which all reflect in detail. This formative process, with origin in Japan and now used all over the world, has been shown to promote the development of teachers' (Richit et al., 2021) and prospective teachers' (Larssen et al., 2018) knowledge. There are many studies about the development of lesson study participants in their pedagogical content knowledge, but only few on the development of their mathematical knowledge (Ponte, 2017). Some research in this regard, such as Vieira et al. (2022) and Leavy and Hourigan (2018), shows that the features of lesson study make it possible to deepen prospective teachers' mathematical knowledge. However, it is not clear what prospective teachers can learn in the field of mathematics and how lesson study supports this learning.

Thus, this study aims to understand how lesson study may promote the development of mathematical knowledge in prospective primary teachers. To achieve this aim, we seek to address two questions:

1. What aspects of mathematical knowledge do prospective teachers mobilize or develop during lesson study?
2. How does prospective teachers' mathematical knowledge develop during lesson study?

PROSPECTIVE TEACHERS' MATHEMATICAL KNOWLEDGE

Teachers need different types of knowledge to teach, including content knowledge, i.e., mathematical knowledge and pedagogical content knowledge (Shulman, 1986). Research on prospective teachers' mathematical knowledge allows them to identify aspects and characteristics that they need to develop and that should be considered in prospective teachers' preparation.

One of the most well-known models of mathematics teachers' knowledge is the *mathematical knowledge for teaching* (MKT) by Ball et al. (2008) with two main domains–mathematical knowledge and pedagogical content knowledge, each one with three subdomains. For mathematical knowledge, this model considered common content knowledge, specialized content knowledge and horizon knowledge. Whereas common content knowledge was regarded as the mathematics knowledge that is learned in mathematical courses in many courses and is used by teachers and in other professions, specialized content knowledge was conceptualized as the mathematical knowledge used for teaching. Horizon knowledge had to do with the relationships between different domains of mathematics across the curriculum.

Carrillo-Yañez et al. (2018) sought to improve this model by modifying its categories. They presented the model *mathematics teacher specialized knowledge* (MTSK) that, as MKT, considers two main fields of teacher knowledge–mathematical knowledge and pedagogical content knowledge–adding beliefs as a central category. Regarding mathematical knowledge, which is the focus of this study, the MTSK model considers three subdomains: *knowledge of topics* (KoT), *knowledge of the structure of mathematics* (KSM), and *knowledge of practices in mathematics* (KPM). KoT is knowledge about mathematical content and its meanings and includes procedures, definitions, properties and foundations, representations, phenomenology, and applications. KSM is knowledge of connections between mathematical topics that make it possible to recognize mathematical structures as an integrated system of elements, including connections based on simplification, connections based on increasing complexity, auxiliary connections, and transverse connections. Finally, KPM refers to knowledge about "doing mathematics", including communicating and reasoning mathematically, knowing how to use definitions, establishing relationships, using representations, justifying, generalizing, and exploring mathematical situations.

When prospective primary teachers begin their initial teacher education, they already have some knowledge of the content they will teach, which they need to develop and deepen during initial teacher education (Piñeiro et al., 2022; Stuardo-Aguayo et al., 2025). Therefore, initial teacher education programs contain subjects related to various mathematics domains. Seeking to map prospective teachers' mathematical knowledge, in the 2011/2012 academic year, a study was carried out in Portugal (Serrazina et al., 2014), involving 268 participants in the first three years of study. The results were alarming, showing that the prospective teachers had limited content knowledge about topics of the first years of school, including in numbers and operations. This study made teaching educators reflect on how they could develop and deepen

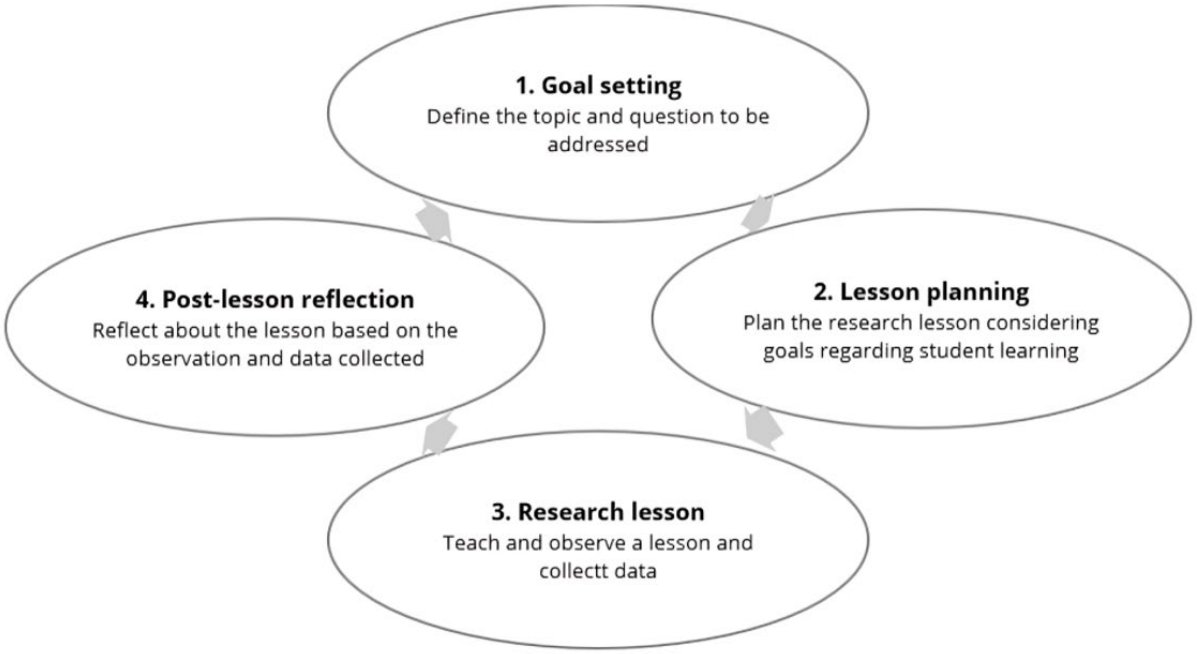

Figure 1. Lesson study cycle based on Murata (2011)

the mathematical knowledge of these prospective teachers. Serrazina et al. (2014) mention the importance of clarifying and deepening the conceptual knowledge of prospective teachers, developing their mathematical reasoning, and capacity to make connections between topics.

In addition to content knowledge, prospective primary teachers need to develop the capacity to promote students' reasoning in mathematics. Therefore, during initial teacher education, they need to develop their knowledge of reasoning and key reasoning processes (Vieira et al., 2020) such as conjecturing, generalizing, and justifying (Stylianides & Stylianides, 2006), as well as other reasoning processes such as exemplifying (Jeannotte & Kieran, 2017).

LESSON STUDY

Lesson study is a teacher education process that aims to improve the teaching and learning of mathematics and has been increasingly used at different levels of education (Ponte et al., 2016). It is based on teachers' collaboration (Murata, 2011) and follows a cycle with several phases (**Figure 1**).

In the first phase, goal setting, the group identifies a problem in students' learning, which guides all the work during the process (Fujii, 2018; Ponte et al., 2016). In the second phase, lesson planning, the group analyses curriculum guidelines, teaching materials, books, and research and professional articles on student learning regarding the selected topic. The participants then plan the lesson in detail by solving, analyzing, and adapting tasks, defining teaching strategies, anticipating possible student difficulties, preparing a whole-class discussion, and organizing the observation of the research lesson (Fujii, 2018; Ponte et al., 2016; Takahashi & McDougal, 2018). In the third phase, the research lesson, one of the participants teaches the lesson planned by the group, and the others observe, taking notes on the students' approaches to the task, their reasoning, and their learning (Murata, 2011; Ponte et al., 2016; Takahashi & McDougal, 2018). The research lesson is a fundamental activity of the lesson study, allowing the participants to see their work put into practice and to observe students' learning, promoting their professional development (Murata, 2011). In the fourth phase, the post-lesson reflection, they discuss and reflect on what they have observed, which can lead to reformulation of the lesson plan. Sometimes, the group decides to teach the lesson again, followed by a new post-lesson reflection.

Given its characteristics and potential, lesson study has been increasingly used in initial teacher education (Tan et al., 2024). Several studies have analyzed the development of teachers and prospective teachers' pedagogical content knowledge during lesson study (Larssen et al., 2018; Cardoso et al., 2023; Ni

Shuilleabhain, 2016). Other studies have shown that lesson study enables participants to develop several aspects of mathematics knowledge (Murata, 2011). For example, Hourigan and Leavy (2019), showed that prospective teachers not only developed a greater awareness of their mathematical knowledge but also recognized the importance of deepening this knowledge, including numerical concepts from the early years school curriculum. Leavy and Hourigan (2018), using the model of Ball et al. (2008), also analyzed the development of specialized content knowledge of prospective teachers in a lesson study. They concluded that the participants developed knowledge about the complex relationships between early numerical concepts that contribute to the development of robust understandings of numbers and how to identify the nature and origin of mathematical errors. Leavy and Hourigan (2018) state that the collaborative work of reflection and the anticipation and analysis of the students' answers contributed to this development. Tepylo and Moss (2011) also looked at the development of the mathematical knowledge of the participants in lesson study. Their study showed that, during the lesson, the participants needed more in-depth mathematical knowledge, distinct from the step-by-step procedures used in an assessment, to see the students' different solving strategies. Finally, Vieira et al. (2022), in a study with prospective teachers, highlights the development of mathematical knowledge and emphasize the contribution of reflection and classroom research.

RESEARCH METHODOLOGY

Participants and Context

The lesson study was carried out in a Portuguese teacher education institution, in 2021, during the last semester of the combined program of pre-school education and primary school, integrated in a *didactics of mathematics* course. This course is mandatory and is focused on aspects related to pedagogical content knowledge for teaching mathematics in primary school. We chose this course because

(1) the prospective teachers had already attended all the mathematics courses of the initial teacher education program,

(2) they were doing supervised teaching practice, making it possible to teach a research lesson, and

(3) the teacher educators could participate in the lesson study.

During this part of the teacher education program, the prospective teachers had weeks divided into two parts: for the first three days, they were in the practice context, in an internship, working with a grade 4 class, under the supervision of the cooperating teacher who taught the class and also supervised by a teacher educator who guided their planning of lessons and observed some of their classes; on the other two days of the week, they had classes at the teacher education institution.

Two teacher educators participated in the study, Miriam and Diana (pseudonyms), who were already familiar with lesson study and were interested and available to collaborate in this study. They had meetings with the researcher (first author) to define the structure of the lesson study and the necessary adaptations. Miriam was the teacher educator who taught the didactics course and led the sessions for the whole class. Diana was the teacher educator who supervised the prospective teachers in the practice context and led two sessions with the group of prospective teachers who participated in this research.

During the lesson, 35 prospective teachers worked on the Didactics course in small groups. The group supervised by Diana (Maria, Isabel, Jennifer, and Julia, pseudonyms) was selected to be observed by the researcher. The prospective teachers had no experience of teaching mathematics and, considering that in Portugal students can choose different paths in secondary education, these prospective teachers had different high school education backgrounds and relations with mathematics. While Maria studied in a science and technology path, with a strong mathematical education, Isabel completed professional education, taking mathematics subjects related to the professional strand. Jennifer and Julia concluded the languages and humanities strand, with Jennifer studying the course mathematics applied to the social sciences, while Julia chose to study no mathematics at all.

During their teacher education program, the prospective teachers attended, in addition to the didactics of mathematics course, several courses that are part of the mathematical preparation, namely: mathematics, culture and reality, numbers and operations, geometry and measurement, statistics and probability, patterns

and algebra, and topics in discrete mathematics, as well as an elective course that could be fundamental concepts of mathematics or materials for mathematical experience.

Lesson Study Structure

The structure and the adaptations of the lesson study to the context took into consideration the aim of using this formative process in initial teacher education in the future. Thus, in the meetings held in advance between the researcher and the teacher educators, they decided to integrate the lesson study into a course. The sessions were structured, and they decided that the prospective teachers would work in small groups, keeping the pairs of the practice context. During the sessions with Miriam, the class of prospective teachers was complete, so the teacher educator circulated around the groups to guide the work and discussions. In the sessions with Diana, only the four prospective teachers were present, so the teacher was discussing the whole time with this small group. Thus, they could carry out the research lessons in their teaching practice, both in a 4^{th} grade class. The researcher assumed the role of participant observer, and in the lesson study sessions, she directed the questions the prospective teachers asked her to Miriam, to not interfere with the normal work process of the course.

In the lesson before the lesson began, Miriam explained what the lesson study was to the prospective teachers and gave information about the process and schedule. The lesson study had 11 sessions, between 60 and 120 minutes long. As there was only one teacher in the room, the groups worked on their own and Miriam went around the groups. To guide the work, at the beginning of each session, the teacher presented the goals and, at the end, brought the whole class back together for final comments and discussions.

In session 1, each group could choose one or two tasks, given that they were internship pairs in different contexts. In the group under study, the prospective teachers selected a task on the topic of numbers and operations. In session 2, the groups solved the task, identified possible students' solutions, anticipated student difficulties and teacher actions to overcome them, structured the lesson, prepared the whole-class discussion, and thought about the lesson conclusion. In session 3, the groups finished the lesson plan and discussed the observation to carry out. Then there was a whole-class discussion about planning of observations. In the end, as the prospective teachers had not finished their lesson plan in the sessions, they decided to hold an extra session on their own. In session 4, they discussed questions to ask the students and finalized the planning. They constructed the statement to give to the students and anticipated difficulties. In session 5, which was also not initially planned, the prospective teachers met with Miriam to clarify some questions about mathematical reasoning and reasoning processes. In session 6, the prospective teachers met with Diana to review the lesson plan and discuss the goals, the task, and the flow of the lesson.

Session 7 and session 8 were the research lessons. The four prospective teachers, corresponding to two previously formed internship pairs, Maria and Julia and Jennifer and Isabel, taught two research lessons, one for each pair. Each pair selected a prospective teacher to teach the lesson, Maria and Isabel, while the other prospective teachers, Julia and Jennifer, observed and took notes. The teacher educators and the researcher did not observe the lessons due to COVID-19 pandemic restrictions. The lessons were taught in the same week, in different classes. Each pair only taught and observed the lesson in the class they knew and where they were doing their teaching practice.

In session 9 and session 10, post-lesson reflection took place in the whole class of prospective teachers. Each group shared their experience. When different groups had the same task, they discussed the differences and similarities of the research lessons. In session 11, the prospective teachers reflected in a small group with Diana about the lesson, how they could improve, and possible adaptations of the task.

Data Collection and Analysis

The research follows a qualitative approach and the interpretive paradigm (Erickson, 1986), adopting a participant observation design (Jorgensen, 1989). The data was collected through video recording of the sessions (S*x*) and recording in a logbook (except for research lessons), semi-structured initial interviews carried out at the beginning of the lesson study (II), in pairs of prospective teachers, and final interviews at the end (FI), one by one, and document collection of the participants' productions during the sessions and their final written reflections (FWR), individual and in group, carried out in the context of the didactics course.

Table 1. Categories and subcategories of analysis used based on Carrillo-Yañez et al. (2018)

Categories	Subcategories	Description
KoT	Procedures	How, when, and why to do something
	Definitions	Characterize and define a concept
	Properties and foundations	Characterize and define properties and fundamental ideas
	Representations	Different representations in which the topic can be considered
	Phenomenology	Relation to content from other subjects and everyday situations, and models and their use
KSM	Connections based on simplification	Connections with previous topics that can be based on more elementary content to make connections
	Connections based on increased complexity	Connections with subsequent topics that can be based on more complex content to make connections
	Auxiliary connections	Connection with other mathematical concepts or in a broader context.
	Transversal connections	Establishment of relationships between different concepts, for example, when different contents have characteristics in common
KPM	Mathematical reasoning	Mathematical reasoning and reasoning processes
	Problem-solving	Recognize and apply the steps of the problem-solving process, the different problem-solving strategies and the correction, difference, and effectiveness of different problem-solving strategies.

Data analysis was done through content analysis (Bardin, 1977) and addresses the three phases of the work: before, during, and after the lesson study. In the first phase, we investigate the prospective teachers' relationship with mathematics. During the lesson study, we analyze the moments in which we observe discussions about aspects of mathematical knowledge. After the lesson study, we investigate the prospective teachers' perspectives about their development of mathematical knowledge. This analysis is based on the categories of the model of Carrillo-Yañez et al. (2018). Initially, we considered the subcategories presented by the authors of the model. Then, we cross-referenced with the data collected, resulting in a slightly modified set of subcategories (**Table 1**).

There was no use of Generative AI in the creation, writing, or revision process of this article. All participants received the guaranty that their participation would be kept anonymous and made an informed consent to be part of this study. This research was approved by the Ethics Committee of Instituto de Educação da Universidade de Lisboa on September 25th, 2020.

RESULTS

Before the Lesson Study

In the initial interviews, except for Maria, who said that "[early years] mathematics is relatively easy for me", the other prospective teachers identified difficulties during their first years of initial teacher education studies:

> Jennifer: I found geometry difficult … The content seemed more complicated than what I'd already learned …
>
> Isabel: Geometry and measurement … I think I've always had a bit of difficulty with mathematics …
>
> Julia: It was all to do with numbers … Numbers and operations were like WOW … Mathematics was the big problem (II).

Maria indicated no difficulties in mathematics. While Jennifer and Isabel said they had more difficulties with geometry, Julia said that numbers and operations were a very challenging domain for her. Regarding the two last years of initial teacher education, the prospective teachers said that they did not have any difficulties because "there aren't so many exams" (Isabel, II), showing that they evaluate their difficulties through the grades they get. Only Julia mentioned difficulties regarding mathematical reasoning (KPM), stating that concerning this capacity "I felt very lost … I really didn't get it at all … My classmates understood and I didn't" (II). Although the prospective teachers said they did not experience any difficulties in the last years, they mentioned some fears about the supervised practice they were beginning in grade 4:

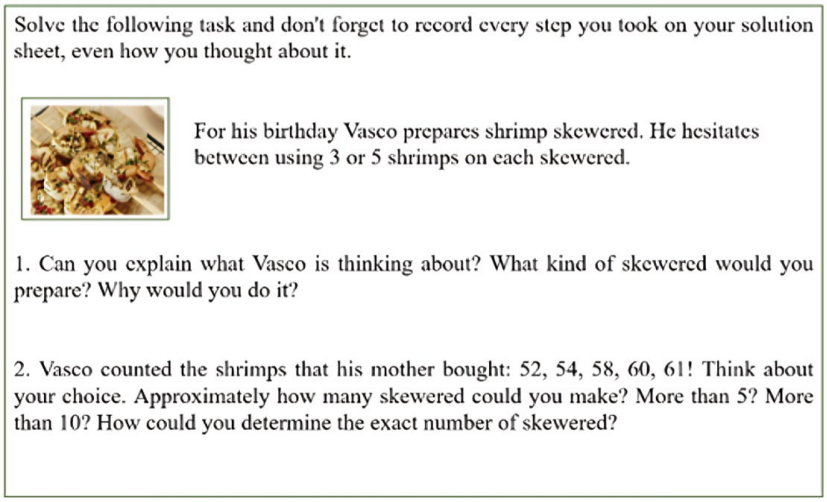

Figure 2. "Shrimp skewers"–Task selected for the research lesson (Source: Authors' collected data)

> Jennifer: [Mathematics] is the area that I have more difficulty, so I think it also makes me feel nervous to support learning (II).
>
> Isabel: There may be more difficulties in grades 3 and 4 than in the grades 1 and 2 because the content becomes a little more complex … The content in grade 2 has things that are probably more common in our daily lives. [For grade 4] I'm going to have to read and review to remember (II).
>
> Julia: About division, I understand nothing of it … Not just because I can't remember how it's done, but because the way it's done has changed (II).
>
> Maria: I forgot some of this content … Sometimes I don't remember specific themes or particular ways of doing things … (II).

Thus, the participants considered that they may have more difficulties with mathematical knowledge when teaching at grade 4. Before the lesson study, the prospective teachers indicated that, from their perspective, they needed to develop aspects of KoT, namely definitions and procedures, and, although with less emphasis, KPM, when Julia mentioned mathematical reasoning.

During the Lesson Study

Phase 1 and phase 2. Goal setting and lesson planning

When the prospective teachers were exploring the tasks to select the topic and the task for the research lesson, more specifically when they were reading the statement, they were able to identify the domain, for example, "task about prisms and pyramids" (Isabel, S1) or "painting squares, it is about fractions" (Maria, S1). They used their mathematical knowledge but did not explain how they came to conclusions.

After exploring several tasks, the prospective teachers selected the "shrimp skewers" task for the research lesson (**Figure 2**).

Miriam guided the prospective teachers in identifying the mathematical topic involved in the task. They quickly concluded that "it is about numbers and operations?! … It is about multiplication and division" (Maria, S1), using KSM (Transverse Connections).

After selecting the task, the prospective teachers started to solve it. From the beginning, Miriam left some clues to help them solve the task, suggesting that the prospective teachers look at the number 61 and think about its characteristics, but they did not realize immediately that this number has only two divisors, showing difficulties in KoT (definitions; properties, and foundations). Miriam then gave time and space for the prospective teachers to develop their knowledge through discussion among themselves, leaving only this clue

as a starting point. Maria said, "[we need to] divide 61 by all the numbers you find that are divisible and find out which ones they are" (S2). To do this, they used a calculator:

> Isabel: There isn't … It's not possible … At least as far as I can see … It's not possible to prepare skewers with 61.
>
> Teacher Miriam: And why do you think that's not possible?
>
> Isabel: I don't think you can because there's no number that, when divided by 61, gives a whole result … There's always something left over … It's always 10.5 or something like that … (S2).

Isabel said the number 61 has no divisors, using KoT (definitions; properties, and foundations). However, none of the prospective teachers mentioned the concept of prime number or could justify their statements, showing difficulties in KSM (transverse connections) because they did not make the connection with the definition of prime number, and in KPM (mathematical reasoning), because they could not explain it. The task is not about the definition of a prime number but understanding the properties of the number 61 would help the prospective teachers to reflect on their solving strategies and deepen their knowledge to solve the task.

When the prospective teachers were solving the task on their own, Julia intervened:

> Julia: Girls, regarding the number 61, I was doing the division of the number and every time you divide by any number you always get an approximate number, it's never a given [whole] number.
>
> Isabel: Yes.
>
> Julia: Isn't the number 61 a prime number?
>
> Maria: Yes.
>
> Isabel: I wonder if that has something to do with it.
>
> Julia: Yeah, I think so! Because you can't divide it …
>
> Isabel: We can try it with another prime number, to see if that … If it is like this too.
>
> Maria: What do you mean? Explain yourself. I don't understand.
>
> Isabel: So, if … If … We can do that example with another prime number and see if that number when divided by another number always gives results with decimal points.
>
> Maria: It does because it's a prime number. There it is, it's only divisible by 1 and by itself (S2).

This conversation provides a deeper understanding of the prospective teachers' KoT (definitions) and KSM (auxiliary connections), especially regarding Julia and Isabel. Julia brought the concept of prime number into the discussion for the first time. Maria quickly agreed that 61 is a prime number, but Isabel had questions in KoT (definitions) when she suggested that they should try out with another prime number. So, in this excerpt, we observe that Julia talked about the properties related to the concept, but in a not rigorous way, referring formally to its designation, and Maria defined the concept of prime number, so we witness an evolution of the prospective teachers' KoT throughout the discussion.

When the prospective teachers were solving this task, when they used the calculator to divide 61 by several numbers and looked at the results, Maria divided 61 by 5 and came up with the number 12.2. The prospective teacher took the opportunity to bring up a topic she had questions about:

> Maria: 12.2 … It is like what? 60%? … Or is it 20%? I don't know if it's like that or the opposite.
>
> Julia: What? I didn't catch it.

> Maria: You have the result 12.2, this... This 20% of the shrimps, what part of the shrimp is it? I never know if it's 20% or 80% ... The point is, if we divide by 5, there's this 20 left over. If they wanted to divide the shrimps, they would have to know what these 20 remaining are. That's so they can divide the shrimps, right?
>
> Julia: Huh? I didn't understand.
>
> Maria: If I don't remember, I don't think they [the students] know what this 12.2 is either ... What these 2, or 20, were equivalent to, whatever ... But maybe we won't go there either, right?
>
> Julia: Ya (S2).

Maria asked a question about percentages, which relation to the task was unclear, and none of her colleagues could understand what she meant. The prospective teachers did not strive to understand the issue, seeming to assume that if they did not know, the students would not ask either. This mathematical question that involves KoT (properties and foundations) was not clarified.

Moving on to anticipate possible solving strategies and student difficulties, the prospective teachers focused again on prime numbers:

> Maria: It would be interesting ... To see ... The ... How do you say ... Something about the tables, I don't know ... The ... What the numbers have in common.
>
> Isabel: The regularities?
>
> Maria: Exactly. The regularities between the numbers, because as they're all organized like this [in a table], you can see the regularities. For example, the number 2 is the only even number that is a prime number.
>
> Isabel: Yes. The rest are all odd numbers.
>
> Maria: The numbers they can automatically see that are not prime numbers are those from the easiest multiplication tables, which are 5 and 10 ... Everything that ends in 5 is from the 5 multiplication table, and everything that ends in zero is from the 10 multiplication table (S4).

Maria seemed to be referring to the Sieve of Eratosthenes. She showed an interest in exploring regularities and tried, through KSM (transverse connections), to make connections with other aspects, but stayed quite far away from the aim of the task, which was not about the study of prime numbers.

Since the prospective teachers still had questions about the reasoning processes (KPM, mathematical reasoning), they decided to meet with Miriam. After a general explanation, they analyzed the reasoning processes that can arise in the "shrimp skewers" task:

> Maria: Justification, exemplification ...
>
> Miriam: ... Perhaps it could also be a generalization, it seems to me that it could arise, but as a learning goal, what is most implicit there is the use of exemplification and the use of justification (S5).

Using KPM (mathematical reasoning) that they used to solve the task, the prospective teachers immediately identified justification and exemplification as the processes they used, while to identify generalization, the teacher's help was necessary. Maria referred to exemplifying as a reasoning process, even though it is considered a supporting reasoning process.

In session 6, with Diana, they looked again at the solution of the task. Diana asked the prospective teachers to distinguish the senses of the division operation, but they showed that they did not know about them (KoT, phenomenology):

> Diana: Division in the meaning of measure ... Why not in the meaning of sharing?

Jennifer: I don't understand the difference, teacher.

Diana: What's the difference between a problem with division meaning of sharing or of measuring?

Maria: Division measure, teacher? I don't know what it is.

Jennifer: I don't know either.

Diana: Julia? Is Julia going to help us?

Julia: I also have no idea.

Diana: What about Isabel? Can you help?

Isabel: I don't know either, teacher (S6).

Diana tried to get a prospective teacher to answer the question. However, she needed to explain, using examples, so that the prospective teachers would understand the difference between division in measuring meaning and sharing meaning, helping them to develop their KoT (phenomenology). The teacher also tried to find out if the prospective teachers already knew that 61 is a prime number:

Diana: And will the result ever be 61?

Maria and Jennifer: No.

Diana: Why?

Jennifer: Because it is a prime number ... It's only divisible by itself and by one (S6).

In this session, the prospective teachers already showed an understanding of prime numbers, showing the development of KoT (definitions) compared to the initial sessions. Still in this session, a discussion arose about the number 60 and its divisors:

Maria: It is a multiple of several numbers.

Diana: What are the divisors of 60?

Maria: 2, 3.

Jennifer: 6, 10 ... 3

...

Diana: So, if 3 is a divisor, which other one can I immediately find as a divisor too?

Maria: The number 6? No ... I didn't understand the question, teacher (S6).

Diana wrote down the divisors that the prospective teachers said so far. They showed some difficulty in identifying the divisors and seeing their relations, for example, that $10 \times 6 = 60$, revealing limitations in KSM (connections based on simplification; auxiliary connections). The teacher made the connection that 6 times 10 gives 60 and that they were both divisors and so on. The prospective teachers were surprised by the arrangement of the divisors in pairs. As they realized that multiplying these two numbers resulted in 60, they showed difficulty in recognizing associated divisors of a number (KSM and KoT). However, this discussion allowed the prospective teachers to deepen their knowledge.

In session 6, when preparing for the whole-class discussion, the prospective teachers discussed how to share the students' strategies:

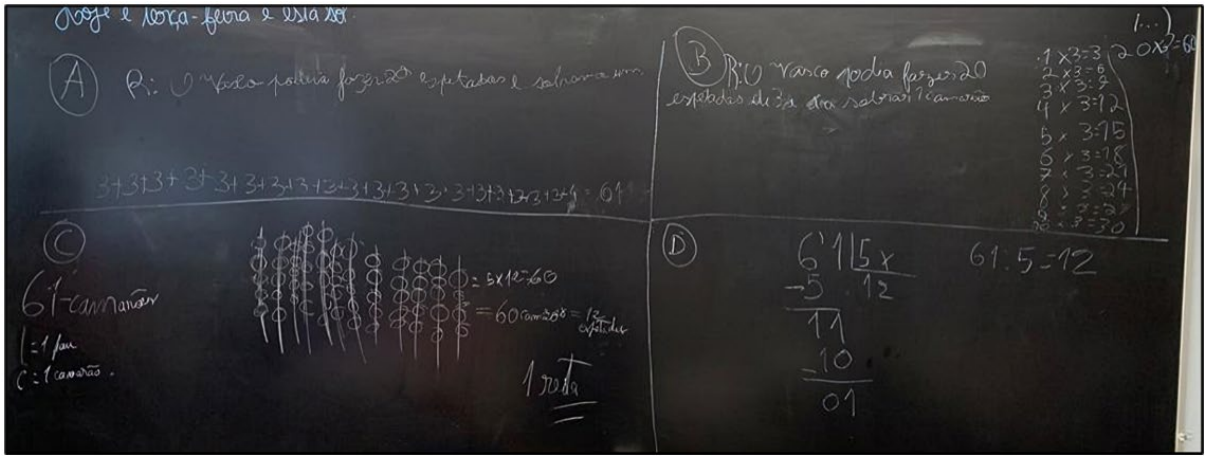

Figure 3. Photo of the board used by prospective teachers Isabel and Jennifer in the final group reflection (Source: Authors' collected data)

> Julia: The operations are sequential … Multiplication and division are inverse, so I think it made sense for them to be together …
>
> Jennifer: We assume that addition is the easiest, then subtraction, then multiplication …
>
> Maria: I would begin with addition. Then, maybe compare addition with multiplication … Because multiplication is the same as addition, only easier, it's a way of getting there more quickly.
>
> Diana: So, you turn the addition of equal parts into a product.
>
> Maria: Yes … The children could use subtraction or not, because I don't see very well how they were going do that.
>
> Diana: So, subtraction, if the skewers have 5 shrimps, for example …
>
> Maria: From 61 they would subtract 5, is that it? But wouldn't that take longer? More complicated?
>
> Diana: But note that it's the process that best translates the action of the shrimps (S6).

This discussion allowed the prospective teachers to reflect on the relationships between the operations to think about their sequence, developing KSM (auxiliary connections; connections based on increasing complexity), and to clarify questions regarding possible solution strategies.

Phase 3 and phase 4. Research lesson and post-lesson discussion

In the final session 9 and session 10, the prospective teachers reflected on the research lessons. For Jennifer and Isabel, the research lesson went as planned, but for Maria and Julia, it was not possible to follow the plan due to time management. Jennifer said:

> Through some of the solutions, it was possible to make this relationship, for example, between addition and multiplication … Then there was one [student] who also did the drawings and through the drawings she also did multiplication … The multiplication solution, I think, was the one we related the most to the drawing solution, where we asked what similarities they found in the two solutions (S9).

In the research lesson, the prospective teachers tried to relate the different strategies, as discussed during the planning phase, showing evidence of their KoT (representations) and KSM (auxiliary connections; connections based on increasing complexity) to relate the different solutions and the addition and multiplication operations that appeared in the students' solutions (**Figure 3**).

Throughout the sessions, several activities led to discussions in which the prospective teachers had the opportunity to deepen their mathematical knowledge. KoT was the knowledge that they mobilized the most and had the opportunity to deepen in the most significant way. However, it was also possible to address and deepen KSM and KPM, although not so frequently.

After the Lesson Study

After the lesson study, when asked about the formative process and the development of mathematical knowledge, Maria, Julia, and Isabel said they had developed and deepened their understanding of reasoning processes (KPM, mathematical reasoning), indicating that in the lesson study "I was able to, let's say, apply … Because in [a previous initial teacher education] class, it was more abstract" (Julia, FI). In the final written reflection, Jennifer also mentioned the reasoning processes used by the students:

> To solve the exercises, the students also used various mathematical reasoning processes, such as conjecture; stating that the number 61 doesn't work because it's odd or because it's only divisible by 1 and itself; generalization by validating the conjecture, 61 being a prime number and the same happening with the number 97; justification and exemplification through the different strategies they used to obtain a result, where they observed that there was always 1 left over (Jennifer, FWR).

Besides identifying several reasoning processes, Jennifer gave examples of their use. The prospective teachers also mentioned that "the exemplification mentioned above supported the other reasoning processes" (Isabel and Jennifer, FWR), although they previously referred to exemplification in a similar way to generalization and justification, showing a deepening of knowledge regarding reasoning processes (KPM, mathematical reasoning).

In the final interview, Isabel added that, while solving the task, she remembered the definition of a prime number (KoT, definitions):

> Researcher: Did you remember the concept of a prime number?
>
> Isabel: No, I knew it … I remember being taught the prime numbers and so on. But it wasn't something that I remembered immediately. We only understood it after a long time solving the task (FI).

Jennifer highlighted KoT (definitions, procedures) and KPM (problem-solving) when she said that "we ended up acquiring some mathematical knowledge, mainly related to the content we were going to develop with the students … With strategies that could be used … Observing the number of different strategies … If it hadn't been for the lesson study, I wouldn't have been as focused as I was" (FI). The identification of different strategies seemed to be a consensual point of view for the prospective teachers, since Maria and Isabel also mentioned it: "[I started to pay more attention] to the different ways of solving a specific exercise, and how to get there" (Isabel, FI).

For the future, Isabel and Julia said they would like to develop a lesson study around "geometry and measurement" (Isabel, FI) and "divisions … Geometry … The question of division because I feel less comfortable … It's the question of not understanding" (Julia, FI). These were the areas in which the prospective teachers mentioned having most difficulties, showing that they considered that the lesson study allowed them to develop their mathematical knowledge.

All the participants emphasized the importance of collaborative work and the significant contribution of the teacher educators to the development of their knowledge during the lesson study. For example, Jennifer said, "It helped that the teacher talked about [the divisors of 60] because from then on we [think], ah, this is probably an interesting thing to talk about during the lesson" (FI). Also, Julia said that "we have different points of view… [so] we were able to come up with different strategies" (FI) and Isabel said that "as we were in a group, I think it was easier because we had everyone's opinion" (FI).

DISCUSSION

In the initial interviews, the prospective teachers mentioned that they still needed to develop their mathematical knowledge regarding some contents and still had difficulties in mathematical reasoning, highlighting weaknesses in KoT and KPM (Carrillo-Yañez et al., 2018). During the lesson, as in the study by Serrazina et al. (2014), difficulties in basic knowledge were observed in the prospective teachers regarding KoT and KSM. For example, although knowledge of prime numbers was not the aim of the task, the participants struggled to identify the number 61 as a prime number, a definition which, although not decisive for solving this task, is essential for the knowledge of prospective mathematics teachers. They were also surprised when they identified the divisors of the number 60, which they should have known from the very first years of school. Despite the many mathematics courses that the prospective teachers had previously attended, they still had questions and difficulties concerning mathematical knowledge, which this formative process helped to address. So, the lesson study promoted situations to overcome these misunderstandings and to develop the prospective teachers' mathematical knowledge, as also found in other studies (Tepylo & Moss, 2011; Vieira et al., 2022). During the lesson study, the prospective teachers mobilized knowledge regarding prime numbers and remembered this concept, in discussions about aspects of KoT. They developed this knowledge while solving the task and developing the lesson plan. The guidance of the teacher educators and the discussions between participants were crucial to this development. KPM was little mobilized in the sessions, only appearing during the solution of the task and, later, when the prospective teachers discussed with the teacher educators the reasoning processes involved in the task. However, it was possible to observe that the prospective teachers developed their knowledge, especially regarding reasoning processes, and mathematical reasoning was mentioned by Serrazina et al. (2014) as an aspect to pay attention to during initial teacher education.

Since the first session, it was clear that discussions between prospective teachers are fundamental not only for clarifying questions but also for reviewing concepts and deepening their mathematical knowledge. However, when the discussions among prospective teachers were not enough to clarify a particular issue, they sometimes asked the teacher educators, as was the case with the reasoning processes. In addition, teacher educators helped the prospective teachers to reflect and deepen their knowledge by establishing relationships, as with divisors of 60.

After the lesson study, the prospective teachers highlighted the importance of this formative process to develop their KPM, particularly regarding reasoning processes. Different from what happened in the study by Hourigan and Leavy (2019), not all of them seemed to give importance to the development of KoT. Although the development of this knowledge was visible throughout the sessions, the prospective teachers did not seem to pay much attention to this development and to the importance of deepening it. Also, they did not mention KSM.

Serrazina et al. (2014) reflected on how they could deepen the knowledge of prospective teachers. Among the aspects to consider, they mention the deepening of KoT and KPM. Lesson study could be an option to deepen this knowledge, since the prospective teachers showed development in different aspects of their mathematical knowledge, although not always with the same depth. The data shows that the lesson study enabled the participants to identify and clarify their mathematical knowledge, an aspect that Serrazina et al. (2014) required new approaches to consider in initial teacher education. It also indicated which moments prospective teachers had the opportunity to deepen their knowledge, develop their mathematical reasoning, and make connections.

The curricular plan of the teacher education program shows that prospective teachers took several mathematics courses during their initial teacher education. Despite the number and diversity of subjects they took, the lesson study made very clear that they showed difficulties and misunderstandings in mathematics. That is, lesson study seems to be a formative process that helps to identify these difficulties and contributes to overcoming them through the activities and discussions that the process promotes. Prospective teachers need to develop and deepen their content knowledge during their initial teacher education, and in this lesson study, it was possible to observe the deepening of the participants' mathematical knowledge. Although it is possible to carry out this development through other teacher education processes, in the lesson study the prospective teachers had the opportunity to develop it simultaneously with another domains of teacher

knowledge (Murata, 2011) in an integrated way with practice and deepening their knowledge by looking for ways to overcome challenges that arose throughout the formative process.

CONCLUSION

The results of this study suggest that the lesson study provided opportunities for prospective teachers to develop their mathematical knowledge about numbers and operations in the three domains indicated by Carrillo-Yañez et al. (2018). They developed KoT, mainly regarding definitions, procedures, properties and foundations, and phenomenology, about the notion of divisors, properties of divisors, notion of prime numbers, and the meanings of division, when they solved and adapted the task and when they anticipated the students' solving strategies. They developed KSM, about different types of connections, when they anticipated the students' solving strategies, looked for connections between them, and thought of a sequence of the different solutions for the organization of the board. They developed knowledge of mathematics practices essentially related to mathematical reasoning when they solved the task and discussed the reasoning processes involved in the task and used by the students.

The lesson study allowed the participants to develop their mathematical knowledge, particularly when they:

(1) carried out activities that required the mobilization of mathematical knowledge,

(2) engaged in discussions that allowed them to deepen their knowledge, clarify questions, and learn about other procedures, and

(3) exchanged ideas with each other and with the teacher educators to improve the lesson plan and to reflect on the research lesson.

Although this knowledge can be developed through other formative processes, the lesson study provides an integrated and collaborative setting for this development, providing enriching experiences for the professional development of prospective teachers. The opportunity to plan-observe-reflect is a feature of this process that enhances the development of knowledge and enables a greater connection between theory and practice.

Tepylo and Moss (2011) have already shown that the development of mathematical knowledge was distinct during lesson study. Leavy and Hourigan (2018) also showed the contribution of lesson study to the development of mathematical knowledge, using the model of Ball et al. (2008), but the focused only on one of the categories of mathematical knowledge of this model, specialized content knowledge. Using a broader framework for mathematical knowledge, based in Carrillo-Yañez et al. (2018), this study analyses in depth which aspects of this knowledge the prospective teachers developed, which lesson study activities promoted such development. Vieira et al. (2022) already highlights the contribution of reflection and the research lesson, whereas in this research the development of mathematical knowledge is visible essentially in the planning phase and also in the post-lesson reflection, although with less evidence. The research lesson was mainly an important contribution to later reflection.

A limitation of this study is the data collection in the research lessons. As the lesson study was carried out during the pandemic COVID-19, the restrictions in place made it impossible for the teacher educators and the researcher to observe the research lessons. Due to logistic constraints, it was also not possible to audio or video record the lessons. Therefore, in each pair, the lessons taught by one prospective teacher were observed only by the other prospective teacher. This influenced the discussion in the post-lesson reflection, as there was limited information shared by the prospective teachers present in the lesson.

This study shows that the development of prospective teachers' mathematical knowledge is a field of research that needs attention. Despite the courses carried out during initial teacher education, the difficulties regarding mathematical knowledge that the prospective teachers had were visible during the lesson study. This research also shows that lesson study is a formative process with potential to develop the different aspects of mathematical knowledge, beyond the courses that the prospective teachers attend during the initial teacher education.

In the future, it would be interesting, in the context of the lesson study, to carry out more studies on the development of mathematical reasoning, as this is an aspect in which the teachers' showed difficulties despite

all the mathematical preparation they had received. In addition, it would be interesting to carry out studies involving other domains of mathematics that the prospective teachers find challenging, and to analyze the development of mathematical knowledge in its different aspects.

This study enabled us to investigate the possibility of using lesson study in initial teacher education and its potential for developing prospective teachers' mathematical knowledge and to deepen how this development may occur during lesson study. It contributes to understanding what aspects of mathematical knowledge the prospective primary teachers mobilize and develop and what activities to carry out at each lesson study moment to develop these specific aspects of mathematical knowledge. The research highlights the importance of the collaborative and reflective nature of this formative process for prospective primary teachers to deepen their mathematical knowledge, through a cycle of plan-observe-reflect. By knowing the potential and characteristics of lesson study that enable the development of mathematical knowledge of prospective primary teachers that experience this process and what mathematical knowledge can be mobilized and developed, teacher educators working within similar programs may identify and carry out similar lesson studies, seeking to improve initial teacher education. A greater understanding of the development of prospective teachers' mathematical knowledge contributes to improve initial teacher education and, consequently, mathematics teaching and learning.

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