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Research Article

Students' mathematics conceptual challenges: Exploring students' thinking, understanding, and misconceptions in functions and graphs

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ABSTRACT

Functions and graphs are fundamental mathematical concepts in mathematics and are vital to helping students comprehend the relationship between variables and other advanced topics in higher-level mathematics. Research has shown that students continually possess misconceptions and inaccurate thinking about functions and function representations. Function concepts such as variations, covariations, and function notations are challenges students face in conceptualizing function concepts in the classroom. Sources of these misconceptions may stem from the way students think about functions. The contributing factor to this problem is the teaching approaches or methods teachers use in mathematics classrooms, which focus on students demonstrating their skill in solving mathematics problems without helping students develop the conceptual understanding of the mathematics they teach. Although function forms the foundation of understanding higher mathematics, students' and teachers' understanding of function concepts appears to be mixed with many misconceptions and wrong assumptions.

The researcher engaged the student in five clinical interview sessions to assess the student's conceptual understanding of selected topics on functions and graphs. The researcher developed and implemented an instructional intervention to strengthen such understanding. A qualitative research method through clinical interview was used to engage a senior high school student in grade 11 in five one-hour meetings to assess the student's conceptual understanding of selected topics under functions. Over the five clinical interview sessions, the interviewer engaged the student's conceptual understanding of topics on functions, such as the meaning of functions, variations, covariations, and function notations.

The researcher developed function assessment questions and validated them by experts for restructuring. All clinical interview sessions were voice recorded and transcribed, and photocopies of the student's worksheets were collected and analyzed quantitatively to support the results and findings of this study. Findings from the study show that students continually develop procedural competencies over conceptual understanding in the mathematics classroom. The student over-relied on the *vertical line test* concept to determine whether a graph diagram represents a function. The student's solutions to the covariation task showed a graphic representation of discrete points with a line drawn through the points to represent a continuous covariation. The designed interventions strengthened the students' understanding and provided a means of testing/validating assumptions about the function concepts and understanding.

Keywords: functions, graphs, concepts, mathematics, thinking, misconceptions

INTRODUCTION

The definition of functions is diversified in many ways, making it difficult to understand the meaning of functions. Functions as ordered pairs or mapping, the univalence definition of a function, functions as an input-output relationship, and functions as an equation or notations are various ways functions have been defined. Moreover, these definitions of functions may also vary at different educational levels. The different representations of function also challenge students' understanding of function concepts. Functions can be represented in various forms, including tables, graphs, and formulas, each offering a unique perspective on the function concepts.

The concept of functions and graphs has been part of the mathematics curriculum for many years, yet students and teachers have different challenges in understanding and teaching this topic. Euler (1748) states, "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities." The definition above means that any analytical expression that, besides having a variable, also contains a constant quantity is a function of the variable. Thus, $a + 5p$ is a function of p. The use of "analytical expression," function as containing a constant, and the vagueness of expressions such as "composed in any way whatsoever" in Euler's 1748 definition of function resulted in a misunderstanding of his earlier definition, leading to providing a more straightforward definition of function as "when certain quantities depend on others in such a way that they undergo a change when the latter change, then the first quantities are called functions of the second" (Euler, 1748, as cited in Ferraro, 2000, p. 2). This definition gives a more precise understanding of how one quantity can determine others when it changes. Euler's refined definition of function means that one item is a function of the other when there is a relation between the two items. According to Thompson and Milner (2018), a function is "a named relation between two sets of elements such that the relation constitutes a rule of association between them" (p. 2). This function definition is consistent with Euler's (1755) more precise definition of a function, which defines a function as a relationship between two entities where a change in one causes a change in the other. These definitions make it abundantly evident that there needs to be a rule of association between these two elements or entities.

Leinhardt et al. (1990), citing the work of Piaget et al. (1968/1977), felt that while graphs have not received much attention or intellectual scrutiny from the educational community until recently, functional relationships have long been acknowledged as significant constructs in the development of abstract mathematical knowledge. Battista and Clements (1991) emphasized the importance of graphs as an effective way of enabling students to picture mathematical concepts, which is significant given the importance of visualization in mathematics. Mathematics interest organizations, such as the National Council of Teachers of Mathematics (1989), have called for the introduction and development of functions integrated with graphs in mathematics curricula in elementary schools. Graphs give a pictorial representation of the abstractness of functions to help students understand functions. Graphs may also give pupils visual representations of mathematical relations and properties, making graphs even more potent in students' comprehension of functions. Representations of functions using computers can also aid in the development of in-depth meaning and an increase in the range of applications of functions. Teachers can foster students' critical thinking skills, acquire a conceptual knowledge of functions, and develop various ways to represent function techniques using computers.

Leinhardt et al. (1990) claim that adding graphs to the middle school mathematics curriculum helps students adequately represent functions. A common approach many educators take when teaching functions and graphs is to regard them as two entirely different symbol systems, algebraic and graphical, that work together to define and develop the mathematical idea of function. Students learning functions in schools have shown that graphs and functions cannot be considered separate ideas in mathematics. Functions now have a more expansive meaning in mathematics education, encompassing their formal definition and the various ways they can be expressed and written. Some common visual representation aids that describe functions include tables, graphs, words, algebraic symbols, and problem scenarios.

Through a series of clinical interview sessions, this research study aimed to investigate a high school student's conceptual understanding and misconceptions related to key topics in functions and graphs, specifically the meaning of functions, variation, covariation, and function notation. The researcher engaged the student to determine the student's conceptual understanding of function, thinking patterns, and underlying misconceptions. A designed instructional intervention was used to address the student's thinking processes and cognitive challenges to provide more effective conceptually driven teaching practices to assist the students in overcoming their conceptual challenges. Identifying students' thinking patterns and underlying misconceptions and implementing targeted educational interventions can help enhance students' conceptual knowledge.

Developing Function Concepts in Mathematics

According to Piaget (1953), children develop the notion of numbers and other mathematical concepts independently and spontaneously to a remarkable degree. Piaget believed that when teachers force mathematical concepts on students before they are ready, students learn the concepts verbally without understanding until their minds genuinely understand them. Piaget's (1971) stage theory on conceptual formation in students earlier did not form the basis for developmental curriculum until mathematics educators abandoned the stage theory before Piaget's general constructs were used to think about developmental learning trajectories for functions and other mathematical topics. To Piaget, cognitive remodeling produces a new type of reasoning and indicates the shift from one stage to the next. Although Piaget's stage theory faced criticism and abandonment in the late 19^{th} century because of contradictions, the basis for the stage theory's development, assimilation, accommodation, and perturbation, is still relevant for conceptual understanding. A conceptual understanding, according to Greeno (1983), is a cognitive object for which the mental system contains processes to accept an object as an input or argument. Greeno's view of developing conceptual understanding stems from the cognitive psychologist's view, while Piaget sees the development of conceptual understanding in the direction of the developmental psychologist. Works from different researchers show that mathematical concepts can have multiple meanings depending on the situation. Mathematical notions, like the concept of function, might have multiple meanings and applications. A function is regarded as a basic mathematical notion that ought to incorporate both pure and applied mathematics and make up a more significant portion of courses like calculus in the mathematics curriculum. As Hiebert and Carpenter (1992) asserted, one of mathematics education's most widely accepted ideas is that teachers must find ways for their students to understand mathematics. Therefore, to assist their students in developing a conceptual grasp of functions, mathematics teachers' knowledge of functions must be founded on an overall idea of function knowledge.

Sfard (1992) emphasized the dual character of mathematical notions as process and object to understand them. Sfard claims that the primary source of the function notion is the computational procedure that begins with a number x and computes a resultant value y. The level of comprehension of the concept of function, however, is to consider a function as a process that associates elements in a domain with elements in a range. Thompson (1985) believes that constructing a function as a conceptual entity is an example of the identification process. According to Biehler (2005), in some cases, comprehension may be necessary to think of a function as a process that associates elements in a domain with elements in a range. When teachers have deep, well-reasoned understandings of mathematical concepts, pupils benefit immensely from teaching. According to Thompson and Milner (2018), any design of items that delves into instructors' interpretations of a concept needs to be based on a scheme of meanings that the designers consider to be the target understanding of the concept. Also, the mathematics classroom should encourage students to create meaningful mathematical concepts independently and connect the concepts created to meanings and activities outside the classroom.

Cognitive Challenges or Misconceptions of Students in Understanding Functions and Graphs

The research literature on teaching and learning function concepts in schools over the past 50 years has concentrated on two main themes, according to Dubinsky and Wilson (2013): students' conceptual challenges in comprehending the concept of a function and theoretical perspectives for analyzing what it means to understand the concept of a function. One of the foundational theoretical works for comprehending the concept of function is Piaget's application of the theory of reflective abstraction to linear functions. It is important to note that various factors, including how students conceptualize function concepts and what they define as functions, might contribute to their conceptual challenges in understanding the concept of function. Understanding implies the connections between mathematical concepts, facts, or procedures, claim Hiebert and Wearne (1991). Gaining an understanding of mathematics requires that students be able to apply mathematical ideas in a variety of contexts, even ones they are not familiar with. Making connections involves building and integrating knowledge structures and connecting newly acquired mathematical knowledge to previously acquired knowledge. According to Clement (2001), many students who learn function concepts see function as a single algebraic formula and believe a formula must represent all functions. It is imperative to help mathematics students differentiate between what can and cannot be described as a function. Constant functions, such as the representation of finitely discrete points on a Cartesian plane coordinate system, usually confuse students when interpreting such a representation as a function. Many students also believe a function must represent only two sets with a one-to-one relation (Zaslavsky, 1990). This understanding of function creates conceptual difficulties in students' minds, such that the expression "$f(x) = 8$" is seen as not a function but a location on a graph.

According to Eisenberg and Dreyfus (1994), secondary school math teachers have traditionally focused their courses on using algebraic representations of functions rather than the graphical method, which impedes students' mental development of functions. For example, Dubinsky and Wilson (2013) pointed out that "some students' misconceptions of the concepts of functions are: function is an algebraic term or a formula or an equation, functions should be given by one rule, graphs of functions should be regular and systemic, should contain a constant algebraic form, $y = ax + c$, where c is a constant, is considered as the representation of a function but $y = 4x$ is not a function because it does not contain a constant." As cited by Walde (2017), Sfard (1992) found that students believed that all functions relating to x and y should be expressed as a formula, and those computational formulas could express all functions. For example, Tall and Bakar (1992) found it interesting "why most secondary school and university students did not regard $y = 4$ as a function because it does not depend on the value of x but regards $x^2 + y^2 = 1$ as a function because it is a familiar and contains variable x and y". Again, function representation concepts are seen as one of the challenges students encounter in comprehending the concept of function. Connecting all functional representations to their definitions is critical in helping students comprehend the concept of function, as this affects how they are understood. According to Thompson (1994), various function representations, including equations, tables, and graphs, should be viewed as a single thing that represents a single object function rather than as distinct entities. Helping students correctly represent the concept of function would help solve many of their challenges. Graphical representation of a function is another challenge students face in understanding the concept of functions.

According to Hitt (1998), students find it difficult to connect various representations of functions, such as formulas, graphs, diagrams, and word descriptions, understand graphs, and work with function-related equations. The review of works of literature on how students' conceptual challenges with the idea of function are related to their incapacity to employ representations like graphs. The conceptual challenges of students in the early stages of learning functions and graphs may fall into one or more categories: *Misunderstanding of the meaning of function, Coordination of axes, Function notation, and Representation and interpreting graphs of functions*. Understanding students' anticipated challenges clearly will help structure a clinical interview.

METHOD

A senior high school student in grade 11 was engaged in clinical interview sessions for five one-hour meetings to assess the student's conceptual understanding of selected topics under functions. The researcher developed assessment questions covering the meaning of function, variation, covariation, and function notation. The questions were validated by experts for restructuring and comments to make them appropriate in helping the student delve into deep thinking, which the researcher sought to investigate. After the student has worked on the questions, the researcher develops and implements an instructional intervention to strengthen the student's understanding.

Clinical Interviews

Clinical interviews are used in mathematics classrooms to determine how well students comprehend, grasp, and solve mathematical problems (Ginsburg, 1997). This qualitative study examined the transcribed voice recordings of the clinical interview sessions and the student worksheet data. This clinical interview allows teachers to explore students' thinking and uncover their basic thinking processes. Unlike traditional testing, clinical interviews concentrate more on students' thought processes than procedural fluency. Clinical interviews offer valuable opportunities for interviewers to delve deeper into students' understanding using probing questions. The data analysis session covers the clinical interview sessions on the meaning of functions, variations, covariations, and function notation.

Data Collection

Each clinical interview session sought to discover how the student interpreted the question, thinking, and a means of validating assumptions about the concept of functions and the student's current understanding of it. All clinical interview sessions were voice recorded, and photocopies of the student's worksheets were collected as evidence for data analysis. The voice recordings were transcribed after each interview session. Data from all the clinical interview sessions were analyzed quantitatively to support the results and findings of this study. Pseudonyms (Gabedeh and AGI) were used to represent the student and the interviewer in the transcription of the interview data to ensure anonymity.

Data Analysis

This qualitative research study's data analysis was based on an interpretive methodology investigating a high school student's conceptual understanding and thought processes about functions and graphs. Verbatim transcriptions of the video recordings were made to ensure accuracy. The researcher critically analyzed worksheet data to identify the student's patterns of reasoning, misconceptions, and signs of conceptual change. To strengthen the validity of the analysis, constant comparative methods were used to identify patterns in the student's responses across different sessions to track new or persistent misconceptions. Using visual interpretation techniques, the student's graphical representations were also evaluated for accuracy and alignment with the conceptual goal. The interviewer asked probing questions to clarify or challenge the student's thinking and direct the direction of subsequent questions.

FINDINGS AND RESULTS

The findings of this study are provided through the interaction and data gathered from the clinical interview. The results are arranged according to the various function topics the researcher engaged the student in during the clinical interview sessions and the student's worksheets.

Misunderstanding of the Meaning of Function

Students may lack deep understanding and mathematical fluency to understand what is and is not a function (Clement, 2001). For example, students may not understand why completing a given table of values represents a given function. The fundamental idea of a function as a univalence may be satisfied by many students accurately matching elements from the domain to a unique element in the range (*an element in the domain must correspond to only one element in the range*); these students may lack the conceptual understanding of univalence and may ignore many-to-one mappings as function while considering one-to-one mappings as functions. Even (1990) emphasized that many students could not differentiate between one-to-many and many-to-one mappings to realize that one-to-many mappings are not functions, but many-to-one mappings are functions. Generally, the student was observed to have a deeper understanding of using the vertical line test approach in determining graph functions. When discussing how to differentiate functions from more general relations, the definition of univalence as a feature of functions is frequently brought up. The overreliance on the "vertical line test" has resulted in students misunderstanding relations and functions. For example, the graph of a circle is misunderstood as a graph of function instead of a relation. On a higher level, however, curves such as the circle can be represented as functions. The vertical line test indicates an easy approach that teachers use to help students learn graphs of functions in many classrooms.

In testing the student's understanding of the meaning of functions, the student was presented with a problem containing diagrams of a domain and range, with arrows pointing elements from the domain to elements in the range. These domain and range diagrams have representations of one-to-one, one-to-many, many-to-one, and many-to-many mappings. Other tasks included graph diagrams of different shapes and forms representing functions and nonfunctions. The task was for the students to determine which diagrams represented a function. For the diagrams containing a domain and range, the student indicated that diagrams A and B are not functions because x-values cannot have multiple y-values. On the other hand, diagrams C and D are functions because each x-value has one y-value, and they pass the vertical line test. In answering the tasks involving graph diagrams, the student constantly drew vertical lines through the diagrams before deciding on the answer choice. The interviewer asked probing questions to understand the student's choice of answers. The initial tasks containing domain and range were discussed before the graph diagrams. The following conversation ensued between the student and the interviewer.

> AGI: Are you sure of your answers, or do you want to make changes?
>
> Gabedeh: I won't make any changes.
>
> AGI: Great. Why do you say diagrams A and B are nonfunctions?
>
> Gabedeh: Diagrams A and B are not functions because x-values cannot have multiple y-values.
>
> AGI: Are you sure? What are multiple are the values?
>
> Gabedeh: In diagram A, 1 corresponds to 1 and 5, and in diagram B, 2 corresponds to –2 and 7.
>
> AGI: In diagram A, I see that 3 and 2 in the domain correspond to only one element in the range. I don't think your explanation holds.
>
> Gabedeh: Because 1 corresponds to 1 and 2 in the range, the whole diagram A is not a function.
>
> AGI: So why are diagrams C and D functions?
>
> Gabedeh: Diagrams C and D are functions because each x-value has one y-value.
>
> AGI: Okay. Let's look at diagrams D. 0 and 5, which correspond to 9 in the range. That seems to be the same explanation you gave earlier, making diagram A nonfunction.
>
> Gabedeh: No, they are different. In diagram A, 1 is in the domain, and it has multiple elements in the range. Diagram D is OK because 9 is not in the domain but in the range, which is a function.
>
> AGI: Okay. In your explanation of diagrams C and D as functions, you added that" they pass the vertical line test." What is the vertical line test?
>
> Gabedeh: We draw the vertical line test to pass through the x-axis, and if it corresponds to only one y-value, then it is a function.
>
> AGI: Wow! That is interesting. But since you did not draw any lines in diagrams A, B, C, and D, how did the vertical line test work?
>
> Gabedeh: I did that in my head because I know if each x-value has one y-value, they pass the vertical line test.

Figure 1 provides captions of the student's explanation for functions and nonfunctions.

Probing the student's understanding of function, the interviewer realized that the student had a strong understanding of the univalence of function as defined by Even (1990). Further attempts by the interviewer to see if the student would change her answer choice did not work. The student could determine which graph diagrams represented a function and which did not. Interestingly, the student used only the *vertical line test* concept to determine which graph diagrams represented a function. In further questioning, the interviewer noticed that the student recognized the values on the x-axis representing the domain and the values on the y-axis representing the range. The student understood that labeled graphs with x- or y-axis markings have no bearing on whether a graph depicts a function. Again, the student could show that a graph of a function does not represent a single pattern, such as a linear pattern. **Figure 2** and **Figure 3** provide situations where the student used the vertical line test to decide on the answer.

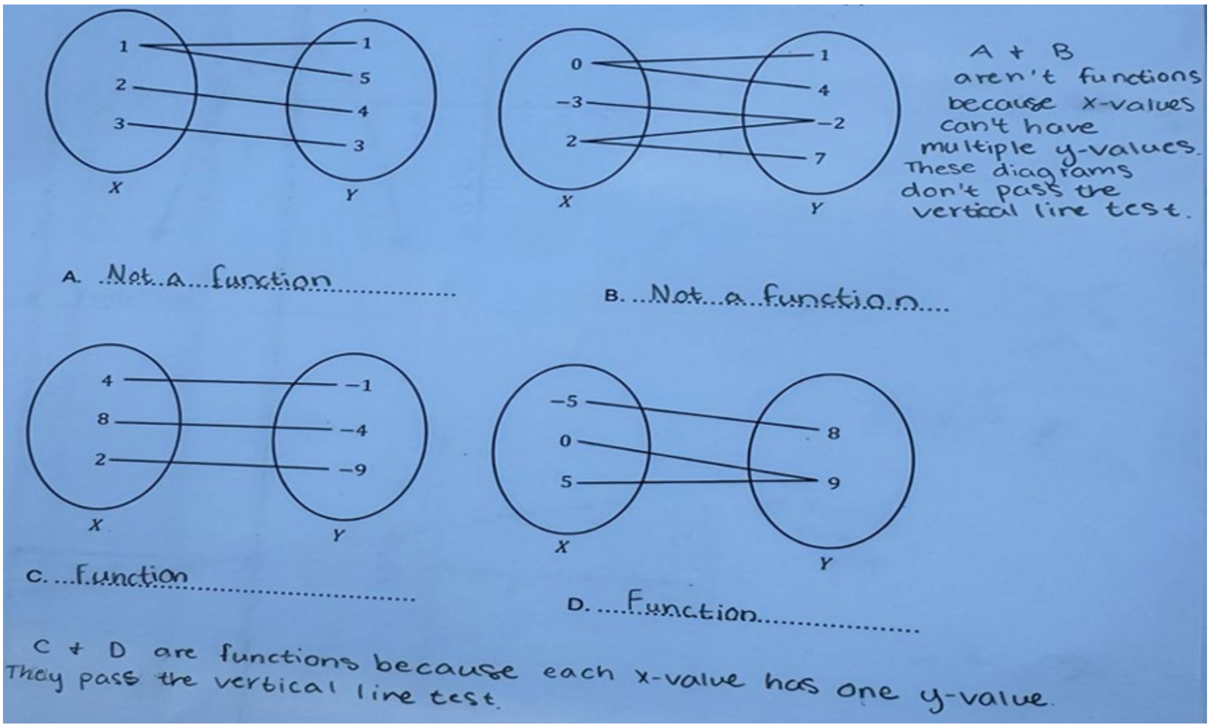

Figure 1. The student's reasons for deciding an answer (Source: Author's clinical interview data)

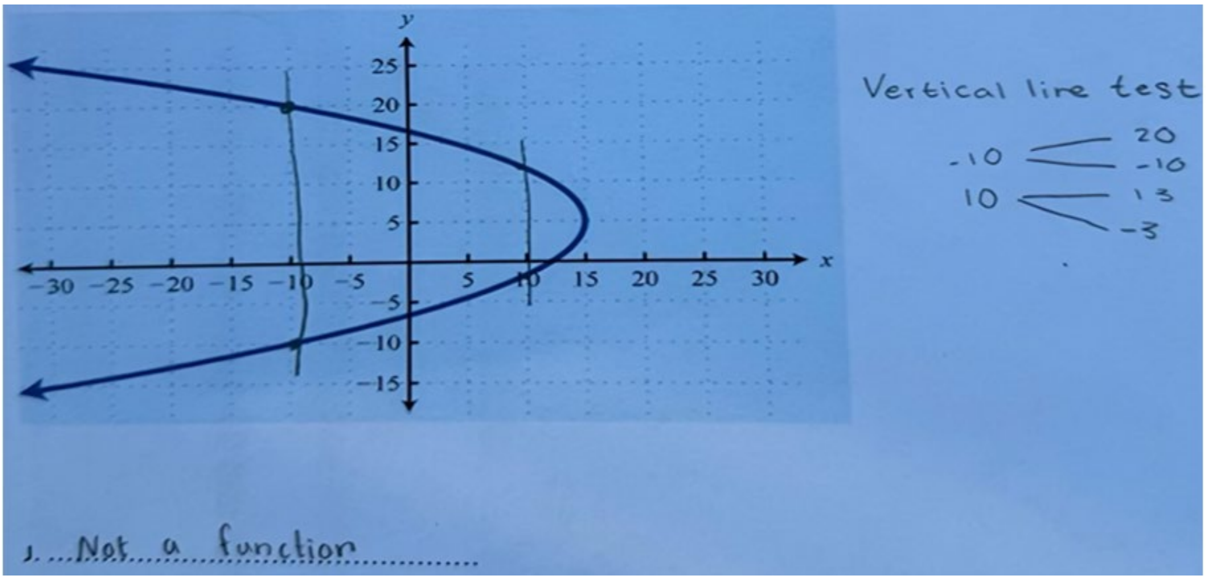

Figure 2. The student's application of the vertical line test (Source: Author's clinical interview data)

The diagrams in **Figure 2** and **Figure 3** show that the student could draw links between all x-values and their corresponding y-values. It is worth noting that the student did not have any challenge applying the vertical line test to graph diagrams that do not show points or graphs in which the points are not connected with drawn lines. Not only does the student know the vertical line test, which can be a superficial skill, but she can relate it to other representations of functions. She has a very sophisticated understanding. One common technique that mathematics teachers use to teach their students to solve univalence function problems is the vertical line test. Students should exercise caution as an excessive dependence on the vertical line test might lead to the misperception that a circle's curve is not a function because it fails the "vertical line test." However, curves such as the circle can be represented at a higher level as functions (Even & Bruckheimer,1998).

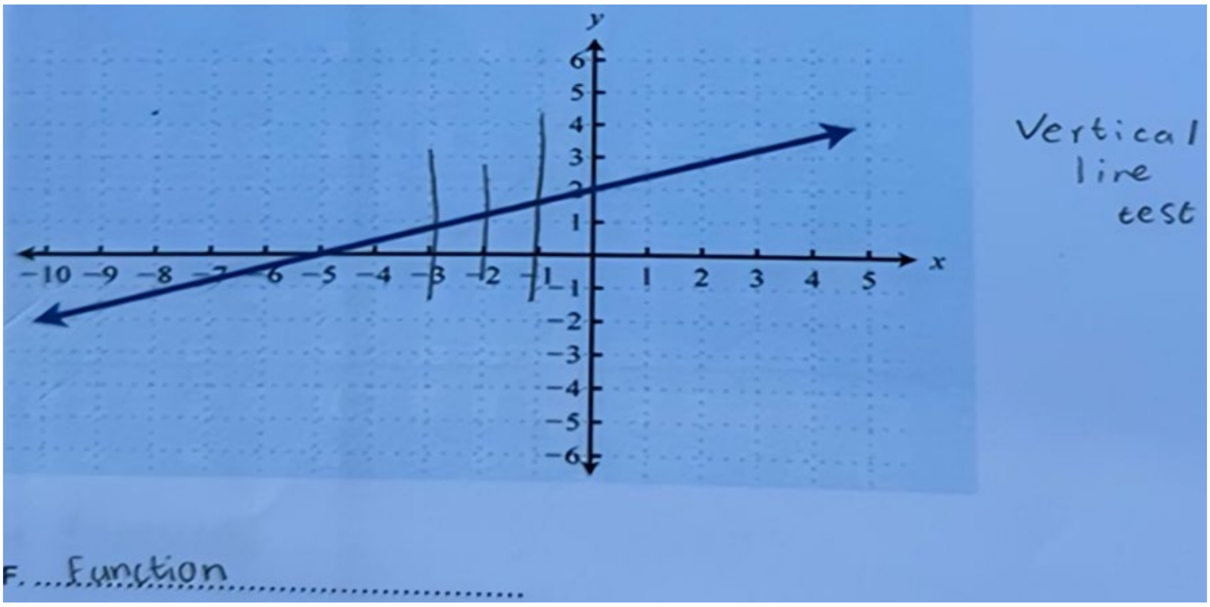

Figure 3. The student's application of the vertical line test (Source: Author's clinical interview data)

Coordination of Axes, Representation, and Interpreting Graphs of Functions

Many students find it challenging to comprehend how the axes in a graph are coordinated, mainly when working with Cartesian coordinates. For example, a student may represent both items in the coordinate (0, 3) on the y-axis or x-axis, leading to a wrong representation of the function or an error in calculations such as finding the function's slope. Understanding which variable, the x-axis and y-axis, represent how changes in one affect the other, and how to represent coordinates in the Cartesian plane can be complex for students. Literature has pointed out that variation and covariation are pivotal for students to build a deeper conceptual understanding of other function components. Castillo-Garsow (2010) explained that students think variationally when envisioning a quantity's value varying discretely. When students have conceptual reasoning of discrete variation, they can differentiate between variation and covariation. For instance, children who cannot reason with discrete variation may imagine that the value of a covariational variable varies in discrete ways, such as "a car has traveled one mile, then two miles, and so forth," without considering whether the car went continuously or between those markers (Thompson & Carlson, 2017). Castillo-Garsow (2010) suggested that students must first develop variational thinking to form the foundation upon which covariational reasoning is developed. Saldanha and Thompson (1998) are of the view that students think covariational when they can hold in mind sustained images of two quantity values simultaneously. Students' misconceptions of variation and covariation are often seen when graphically representing continuous and discrete functions.

The graphic representation of various function types, such as exponential, quadratic, and linear, is a major problem in students' understanding of functions. Graphing functions may be complex for students at an early stage of learning mathematics, as they need to comprehend how variations in coefficients and exponents impact the graph's shape and location. Students may find it challenging to interpret the information presented in graphs accurately. Understanding concepts like slope and intercepts requires students to connect the visual representation of data with underlying mathematical concepts. Functional relationships or graphical representations of data functions are essential when teaching and studying function ideas. However, studies have demonstrated that using graphs or pictures to illustrate functions is difficult for pupils and can quickly result in misunderstandings. Literature from researchers (Bell & Janvier, 1981; Clement, 1985; Leinhardt et al., 1990; Nitsch, 2015) has identified function representation misconceptions such as graph-as-picture misconception, slope-height confusion, or interval-point confusion.

According to Nitsch (2015), not all students' mistakes encountered in working functions are misconceptions, and not all misconceptions are immediately apparent, but they must be exposed eventually. Amidst all these conceptual difficulties, students face in functions and graphs tasks discussed above, students can construct their conceptual understanding of the task. Teachers must tap into the students' thinking or

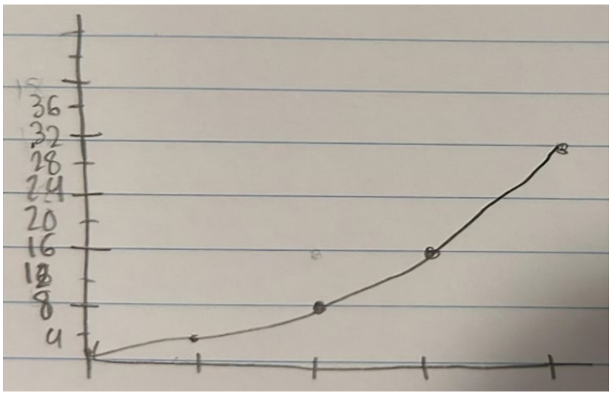

Figure 4. Student's final representations of word problems on variation (Source: Author's clinical interview data)

reasoning to develop strategies that will lead the students to the correct conceptual understanding. von Glasersfeld (1995), drawing on Piaget's genetic epistemology, clarified that from a constructivist standpoint, knowledge is created in a person's mind when they form acts, have new experiences, and consider the results of those activities. Assimilation, accommodation, and schemes formed from perturbations are fundamental components that define what is meant by individual student meanings according to Piaget's genetic epistemology. These concepts form the foundation for how people create new meanings, alter preexisting meanings, and incorporate them into their existing comprehension. Usually, how students understand tasks is not the same as what the teacher or the researcher intended. Teachers must investigate how students interpret tasks to create understanding from the students' thinking through students' interpretations and analysis of their work.

Since covariation is fundamental to comprehending the rate of change and functions, much research has concentrated on students' covariational thinking. According to Castillo-Garsow (2010), for students to construct covariational reasoning, it would be important to construct variational reasoning, which forms the foundation upon which covariational reasoning is developed. The student was presented with word problems on variations—values varying discretely and covariation—questions that focus on the student's ability to hold in mind a sustained image of two quantities' values (magnitudes) simultaneously (Saldanha & Thompson, 1998). The task was for the student to provide a graphical representation of the word problem tasks on the Cartesian plane. The student's solution to the variation task (provided in **Figure 4**) showed that the student did not have much difficulty graphically representing the word problem on the Cartesian plane. The student's graph diagram showed the application of what Castillo-Garsow (2010) termed "smooth continuous variation." Thus, the student thinks of variation of a quantity as values increasing or decreasing by intervals, and knowing that within each interval, the quantity value varies smoothly and continuously. For example, in the word problem task 1, the student could correctly find the variation between the cost of the car in the year 2020 and other years. The student determined that the car's price keeps increasing year after year and presented these changes correctly on the Cartesian plane using points. Again, the student understood what Castillo-Garsow (2010) termed "chunky continuous variation"—thus, the student thinks of variation of a quantity as values changing by intervals. For example, the student could tell how much variation exists in the price of the car between years and from year to year.

Shockingly, despite the student's understanding of plotting discrete coordinate points on the Cartesian plane in our previous clinical interview session, the student drew a line to join all the points plotted from the information from the word problem tasks. For example, problem one required the student to graphically represent the different prices of a car, whose price doubles year after year. These yearly price changes are discrete and do not happen every day throughout the entire range, but after the student has successfully plotted these yearly prices, a line was drawn to connect all these points. **Figure 4** and **Figure 5** show the student's final representations of word problems on the Cartesian plane.

Probing questions on why the student drew a line through the points revealed that the student had fallen on a procedural competency—skills that all points drawn on a Cartesian plane must be connected with a line

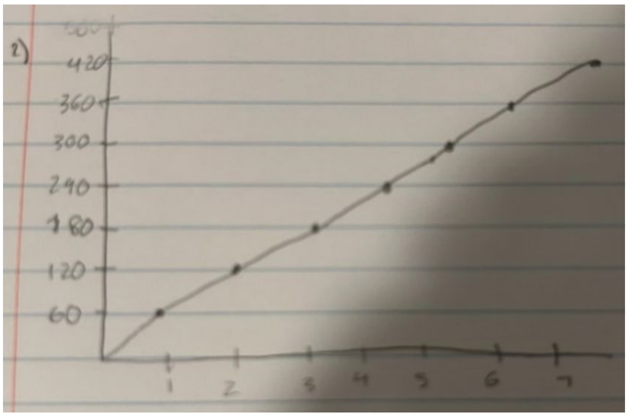

Figure 5. Student's final representations of word problems on variation (Source: Author's clinical interview data)

to create a pattern, even though the student answered that these points are discrete. The conversation below ensued between the student and the interviewer:

> AGI: Great. I can see that you have marks on the x-axis, but you didn't label them.
>
> Gabedeh: Oh. Those are the years.
>
> AGI: Okay. I see a mark at the intersection of the x and y axes. Is that point zero?
>
> Gabedeh: That will be 2,000 dollars for the year 2020.
>
> AGI: I have a question. Why did you draw a line through all the points to connect them?
>
> Gabedeh: I feel like there should be a pattern, and connecting the points will give that pattern.
>
> AGI: So, do we always have to connect points to give us a pattern?
>
> Gabedeh: Hmm. I think not really, but we can do that to give us a pattern.

In both tasks, the student drew lines to connect these points, though these problems involve discrete variation. The intervention focus was to help the student conceptually reason about discrete variations or points. The interviewer provided a covariation scenario to help the students realize that discrete variations do not necessarily need to be presented with a line of connected points.

The student's solutions to the covariation task showed a graphic representation of discrete points with a line drawn through the points to represent a continuous covariation. Even though the graph presented by the students shows hours and distance traveled by a car moving simultaneously by showing the movement with a line, the students still made discrete points on the line. The line drawn by the student shows that the student understands the car traveled a certain distance continuously and that time also passed continuously without breaks; however, she represented the time and the distance in chunks using points. The interviewer engaged the student in conversation to ascertain why there are plotted points on the line as below:

> AGI: Nice graph you have there. You have a similar graph to your graph in our previous session?
>
> Gabedeh: Yes. I. I used only points at first but I joined the points because it is continuous.
>
> AGI: What is continuous? The graph?
>
> Gabedeh: Last time, you talked about continuous graphs, and I think this graph should be continuous.

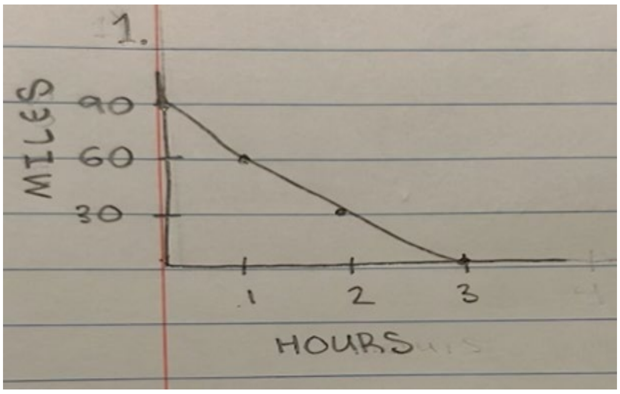

Figure 6. The student's final representations of word problems on covariation (Source: Author's clinical interview data)

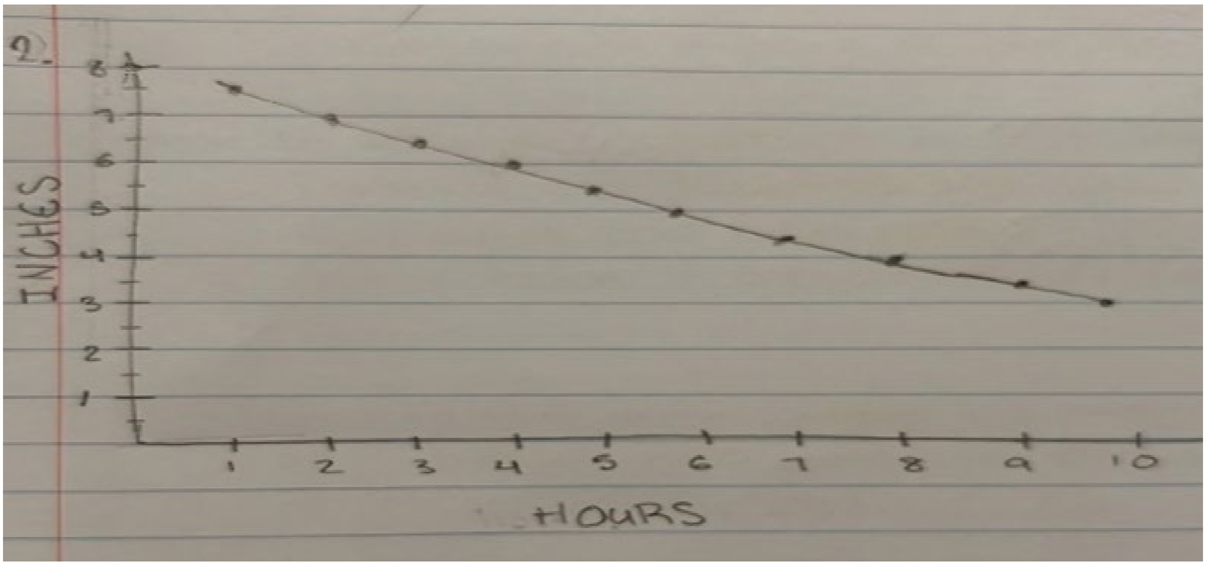

Figure 7. The student's final representations of word problems on covariation (Source: Author's clinical interview data)

AGI: Oh Okay. And we talked about discrete points too, right? I can see you have points on the line. Meaning your graph is both continuous and discrete?

Gabedeh: The question says the car travels, I think, 30 miles per hour, so the points show every hour and the distance.

The conversation shows that the student has conceptualized the rate of change as happening as a chunky continuous covariation. Once more, the student graph demonstrates Castillo-Garsow's (2010) concept of covariation as a coordination of values, which aims to create discrete pairings by coordinating the values of one variable with another. The student's graph for the two questions is displayed in **Figure 6** and **Figure 7**.

Figure 6 on the car problem presented the student's problem, which the student interviewer missed during the clinical interview session to address: the graph slopes down instead of up. Since the problem asks for "a graph to represent the distance traveled by car to reach the bottom of the hill," the student's representation of the problem on the graph should show an increase in both cases because the distance traveled is increasing. This is because the shape of the route has no bearing on the graph. The image provided by the student suggests that the student did not create a graph reflecting the covariance of variables; she may be creating a picture of the path taken by the car.

Function Notation

Many research studies have found that teachers and students struggle with function notations. A study by Thompson and Milner (2018) on functions indicates that US students think that saying $f(x) = y$ is complicated and unnecessary. Introducing function notation, such as $f(x)$ or $y = f(x)$, can confuse students accustomed to thinking of equations in terms of y alone. Understanding that $f(x)$ represents a function and that x represents the input variable can be challenging for some students. The idea of an input and output procedure for learning functions creates a misconception in students' minds that the expression on the left-hand side of function notations represents the function's name for the output on the right-hand side. Even though students know what represents the input variable in a function notation, they hardly recognize the differences between the function name and the input value. Students mistakenly combined the function name (f) and the input variable or value (x) to represent the function name. Students generally misunderstand when recognizing a function notation's different components.

Thompson and Milner (2018), in agreement with Dubinsky and Wilson (2013), revealed through the findings of their studies that students usually have difficulty identifying the components of function notations. Borrowing the idea from Thompson and Milner (2018) on how teachers represent $f(x)$ as a function name instead of the value of a function, the clinical interview session will focus on the extent to which the student will use $f(x)$ as function names instead of the value of the function. A similar question based on Thompson and Milner's (2018) example was developed for this clinical interview session as shown below:

The following are definitions for two functions, $\boldsymbol{a}$ and $\boldsymbol{b}$. This is followed by a third function, $\boldsymbol{c}$, defined in terms of the previous two functions. Place the correct letter in the definition of function $\boldsymbol{c}$ to properly define it in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\boldsymbol{a}(x) = \cos(x+1) \text{ if } x > 1$$

$$\boldsymbol{b}(h) = \sqrt{h} + 5 \text{ if } h > 0$$

$$\boldsymbol{c}(m) = \begin{cases} \boldsymbol{a}(\_) & \text{if } \_ < 1 \\ \boldsymbol{b}(\_) & \text{if } \_ > 0 \end{cases}$$

From the student's answer to the task, the student placed the same input variable from functions $\boldsymbol{a}$ and $\boldsymbol{b}$ into function $\boldsymbol{c}$. The student believed that since the third function, $\boldsymbol{c}$ is defined based on functions $\boldsymbol{a}$ and $\boldsymbol{b}$, the student must reuse the same input variables from functions $\boldsymbol{a}$ and $\boldsymbol{b}$ in function $\boldsymbol{c}$. The student had the misconception that the input variable, placed in parentheses, forms part of the function name. The student thinks of $f(x)$ as a function name; hence, any function defined based on such an initial function should also use the same name. Even though the student understood from prior knowledge that the input variable can represent different values from the domain, resulting in outputs in the range, the student failed to represent such understanding in the case of a function notation. **Figure 8** represents the student's thinking about the problem.

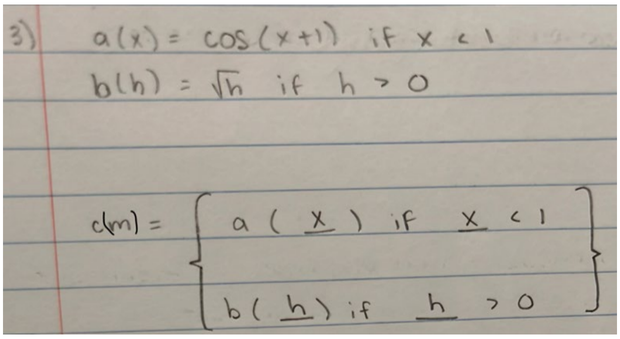

Figure 8. The student's final representations of function notation (clinical interview data)

The student worksheet reveals that the student thinks of $f(x)$ as a function name. The student's diagram is similar to the findings of Thompson and Milner's (2018) work on how teachers represent $f(x)$ as a function name instead of the value, because the student also placed x and h in the blank spaces of function $\boldsymbol{c}$.

DISCUSSION

Students continually possess various misconceptions and inaccurate thinking about functions and function representations. Sources of these misconceptions may stem from the way students think about functions. The contributing factor to this problem is the teaching approaches or methods that mathematics teachers use in mathematics classrooms, which focus on students demonstrating their skill in solving mathematics problems without teachers helping students develop the conceptual understanding of the mathematics they teach in the classroom. Although function is central to understanding mathematics, students' and some mathematics teachers' understanding of functions appears to be mixed with many misconceptions and wrong assumptions. Over the five clinical interview sessions, the student's conceptual understanding of topics on functions, such as the meaning of functions, variations, covariations, and function notations, was assessed. It must be noted that the clinical interview did not look for how accurately and consistently the student answered the question, but rather the thinking that motivated the answers provided, leading to an intervention to strengthen the thinking or help clear the student's misconceptions.

Findings from the clinical interview sessions show how students continually develop procedural competencies over conceptual understanding in the mathematics classroom. Despite its limitations, using the vertical line has become one of the easiest ways for many students to differentiate between functions and nonfunctions. Students have challenges conceptualizing variation and covariation, which is foundational to students' understanding of the rate of change and other higher-level applications of functions. Also, research on functions has revealed that students and teachers have difficulty understanding function notations because function notations are seen as complex and challenging to understand. This complexity in understanding function notation is evident in Thompson and Milner's (2018) research on functions, indicating that US students believe that saying $f(x) = y$ is complicated and unnecessary. In most cases, many mathematics teachers struggle to differentiate between skills and conceptual understanding or misunderstanding. This was evident during the clinical interview session as the researcher used skills and conceptual understanding interchangeably. This may be due to how mathematics is taught and learned in our classrooms, where skills have been the teachers' focus—students following mathematics procedures and reproducing the same procedures.

According to Nitsch (2015), not all mistakes are misconceptions, and not all misconceptions are visible on the surface, but they need to be uncovered in time. This clinical interview has proven that the best way to enhance student conceptual understanding is to help students develop mathematical thinking as the mathematics teacher implements an instructional intervention to strengthen that thinking and understanding. On the other hand, the clinical interview has shown that mathematics teachers have limited their students' conceptual understanding capacities over the years by shifting students' attention to following and reproducing answers to mathematics problems. For example, the student drew a line through discrete points in one of the tasks in the clinical interview. Probing questions showed that the student only drew the line because it had been the routine in most mathematics classrooms. It would be necessary for all mathematics teachers to focus on students' conceptual understanding of the mathematics taught to improve the general understanding of students and teachers in the mathematics education field.

CONCLUSION

In the clinical interviews with a grade 11 student, the researcher examined how the student interpreted and reasoned through the meaning of functions, variations, covariations, and function notations. The findings suggest that while procedural fluency, such as using the vertical line test, was evident, deeper conceptual understanding was often lacking (Even & Bruckheimer, 1998). Despite procedural fluency in many mathematics methods, students often have deep-seated misconceptions about fundamental mathematics ideas, such as univalence, discrete vs. continuous variation, and the true meaning of function notation. For example, the student correctly applied the vertical line test but misunderstood many-to-one mappings as invalid, revealing a limited grasp of the foundational univalence property of functions as highlighted by Even (1990), as one of the challenges students encounter in function and graph problems. This research study exposes the traditional mathematics teaching methods as focusing on procedural fluency rather than conceptual exploration. Procedural fluency limits students' ability to build lasting and transferable mathematical understanding.

Other conceptual difficulties included incorrect assumptions about discrete versus continuous variation and the belief that function notation [e.g., $f(x)$] referred to a function's name rather than its output, as indicated by Thompson and Milner (2018). The findings also demonstrated that students could internalize incorrect understandings that remain unchallenged in traditional classrooms. These conceptual difficulties are similar to previous research findings, such as those of Hiebert and Carpenter (1992) and Dubinsky and Wilson (2013), on how classroom instruction frequently leaves misconceptions unaddressed. However, through guided questioning and targeted interventions, students can begin to reconstruct and refine their understanding. For instance, after the instructional intervention, the student gained a deeper understanding of discrete and continuous data and differentiated between the conceptual understanding and visual procedures, such as connecting points with lines. The student's improvement in thinking and representation after the instructional intervention and probing questioning suggests that clinical interviews can reveal and correct students' long-term misconceptions.

Contribution of the Study

This study contributes substantially to the expanding body of information on mathematics education by highlighting the significance of conceptual understanding in learning functions and graphs. Clinical interviews as a qualitative method provided rich insights into students' thought processes, an area often overlooked in more quantitative assessments. Clinical interviews align with Sfard's (1992) view of mathematical understanding as both a process and an object. The study emphasizes the importance of engaging in mathematical thinking, a significant component of Piaget's (1953) constructivist theories of cognitive development. Furthermore, the study found specific misconceptions such as the overreliance on the vertical line test, the confusion between function notation and variable inputs, and the misapplication of continuous graphing to discrete data. These findings are particularly valuable for mathematics educators, curriculum designers, and teacher educators to develop a rigorous framework for diagnosing and addressing conceptual gaps in student mathematical understanding. By emphasizing the importance of understanding student thinking and designing responsive instructional interventions, this research supports existing recommendations, such as those of von Glasersfeld (1995) and Greeno (1983), to implement more constructivist and student-centered approaches in mathematics classrooms. The study also provides theoretical and practical implications for improving mathematics teaching and learning.

Limitations of the Study

While this study offers valuable contributions to studies on conceptual development and understanding of functions and graphs, it has limitations that should be acknowledged. Notably, the study involved a single high school participant, limiting the generalizability of the findings. Although in-depth one-on-one clinical interviews yield rich qualitative data, they cannot capture the diversity of misconceptions and learning experiences across different classrooms, regions, or educational systems. Additionally, the student selected exhibited above-average reasoning skills, which could skew perceptions of how widespread or severe certain misconceptions are. Again, the researcher served as interviewer and evaluator of the data, which may introduce bias into data interpretation. The absence of additional data sources, such as classroom observations, teacher input, peer comparisons, and other quantitative data analysis methods for triangulation, further narrows the research context. Moreover, while the intervention successfully addressed some misconceptions, the long-term retention and transferability of the student's newly acquired understanding gained during the intervention were not assessed. Lastly, the clinical interview format does not replicate the dynamics of a typical classroom, where peer interaction, time constraints, and instructional scaffolding play vital roles in learning. Future research should include a more diverse sample that involves multiple students and evaluators to explore the effectiveness of interventions in broader educational settings.

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Ethics declaration: The author declared that this study was approved by the school principal of the high school where the student was selected for the study on January 6th, 2025. A student and parent consent form was obtained before the commencement of the clinical interview. The participating student volunteered to participate in the study without any coercion. No sensitivity and identifiable personal data was collected and included in the study.

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Data availability: Data generated or analyzed during this study are available from the author on request.

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