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Research Article

Revealing Implicit Knowledge of Pre-Service Mathematics Teachers in Lesson Planning: Knowledge of Infinity

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ABSTRACT

The study aims to explore how pre-service mathematics teachers reveal their knowledge of infinity during the lesson planning process in the context of limit. Specifically, we adopted the dimensions of mathematics teachers' specialized knowledge, which are related to mathematical knowledge and its relation with teaching. We conducted an exploratory case study design in a two-cycle lesson planning process with three senior pre-service mathematics teachers. The findings indicated that in the first planning process the participants did not use infinity as a way of thinking, rather they focused on paradoxes and potential infinity. After they taught their lesson plan, they started to think about the complexity of infinity not only in the context of limit but also in the concept itself during the lesson planning process. The findings did not cover the knowledge of infinity; rather, they yield important implications for lesson planning process to reveal teachers' knowledge for teaching.

Keywords: lesson planning, pre-service mathematics' teachers, mathematics teachers' specialized knowledge, infinity, teacher education

INTRODUCTION

Though the notion of infinity is not included in the elementary and secondary school curricula explicitly (e.g., MoNE, 2018; NCTM, 2000), it forms the base for many concepts including calculus concepts-limit, derivative and integral (Monaghan, 2001) and it is a concept that develops students' mathematical thinking for many mathematical concepts. However, the concept is misinterpreted not only by students and also by pre-service teachers. Furthermore, the conceptualization of infinity depends on knowledge of the instructors and the context where teaching occurs (Jirotková & Littler, 2004; Monaghan, 2001). Therefore, it can be said that the knowledge and understanding of the notion of infinity is required to teach the calculus concepts, in particular the concept of limit. However, similar with the school curricula, the concept is not sufficiently mentioned in pre-service teacher education programs (Yopp et al., 2011).

What we know about pre-service mathematics teachers' knowledge of infinity is largely based upon the studies that investigate how they conceptualize and understand the concept of infinity (e.g., Date-Huxtable et al., 2018; Kolar & Čadež, 2011; Wijeratne & Zazkis, 2016). Though these studies have investigated pre-service mathematics teachers' mathematical knowledge of infinity in the eyes of conceptualization, they didn't focus on instructional practices including thinking on students' learning, planning a lesson, teaching the concept. This issue has not received as much attention as the studies on pre-service mathematics teachers' knowledge (Montes & Carrillo, 2015).

Figure 1. The lesson study model

This study aims to fill the gap in the research literature on pre-service teachers' knowledge of infinity about reflecting their mathematical knowledge in the concept of infinity. The pre-service teachers were engaged in a lesson study process which was designed for developing their knowledge about the concept of limit. In particular, we focused on the lesson planning stage of lesson study to reveal their implicit knowledge-knowledge of infinity for the concept of limit. In this way, the results of the study aim to shed light on the development of knowledge of infinity in teaching for pre-service teacher education by revealing to what extent pre-service teachers' knowledge of infinity confronts them in the teaching of the concept of limit and what kind of processes they go through.

Knowledge of Infinity for Teaching the Concept of Limit

The historical developments of the limit and infinity concepts can be described as there is no daylight between them (Kim et al., 2005). The starting point of infinity named as potential infinity is observed the first conception of students about the concept of infinity. The tendency to explain the limit as a potentially infinite process that goes on forever and never quite reaches its desired goal is an unavoidable fact (Kidron & Tall, 2015, p. 184). Such a conceptualization can be explained as a meaning of potential infinity as well as dynamic notion of infinity. This fault line relation between potential infinity and limit concept is derived from not only the historical issues (Kidron & Tall, 2015) but also teaching the concept. On the other hand, defining actual infinity as an entity, which is closely related to the concept of sets, revealed another conceptual meaning of infinity. In order to answer students' needs effectively and plan and implement in class tasks, teachers should possess sufficient theoretical knowledge on conceptual meaning of infinity, using appropriate paradoxes and metaphors, and on applications which use the notion of infinity in the concept of limit as well as the practical knowledge on how to implement it (Montes & Carrillo, 2015). Such knowledge is related to foundations and phenomenology of concept, practices in concept, mathematical communication, and teaching strategies.

While some previous studies have considered knowledge of the notion of infinity in the context of limit with working on tasks, examining with tests and textbooks (e.g., Aliustaoglu & Tuna, 2019; Kajander & Lovric, 2017; Montes et al., 2014; Montes & Carrillo, 2015), none have addressed pre-service teachers' knowledge of infinity in preparing contents in a lesson about the concept of limit through a well-designed process for developing their specialized knowledge for teaching limit. Thus, we believe that our results lead to both methodological and practical implications for pre-service teacher education.

Lesson Planning: An Effective Way for Revealing Mathematical Knowledge

In teacher education, how teachers cope with the challenges they encounter is closely related to the knowledge and awareness they develop. One of the ways to improve this knowledge and awareness is learning from study of practice (Zavlavsky, 2008). Lesson study is one of the ways for teachers' and pre-service teachers' learning from the study of practice. (Lewis, 2002). Lesson study is a teacher development model in which teachers work collaboratively to improve their teaching based on students' learning (Lewis, 2002). It can be described as a cycling process including four successive and repeating stages which can occur over a number of weeks (see **Figure 1**). These stages are as follows:

1. setting a learning objective for a topic in the curriculum that students have difficulty in understanding,

2. building a research lesson plan in collaboration that envisages how students would react to the concept by paying attention to elements of their learning process such as the materials, content, trajectories and textbooks on that concept,
3. implementing the research lesson in a real classroom where one of the group members teaches the concept while others record students' reactions and take notes of their thinking processes, and
4. reflecting and discussing on how effective the lesson was in facilitating student learning. This stage might be followed by discussions on how to improve the lesson plan, and the cycle might be applied again to revise on missing points. (Hart et al., 2011).

In planning a lesson, teachers have the chance to think on students' expectations and possible actions, to prepare themselves to students' thinking and to develop not only students' mathematical understanding but also their own mathematical knowledge and mathematical thinking (Smith & Stein, 2011). Lesson planning comprises of thinking on all the aspects of teaching including setting goals for the lesson, formulating appropriate strategies, preparing activities and arranging them with assessment strategies in an appropriate order as well as the knowledge for teaching the related topic (Umugiraneza et al., 2018). Zavlavsky (2008) indicated that knowledge of pre-service teachers is constructed through iterative process by means of prompts of a facilitator and feedbacks from the tasks included in lesson plan. In this context, since lesson planning phase in lesson study includes some critical issues such as constructing the teaching process from beginning to end (Rusznyak & Walton, 2011), it can be said that lesson planning is a tool for solving the complex process of teaching (Umugiraneza et al., 2018) which meets the challenges including adaptability, awareness of similarities and differences in mathematical concepts, conflicts, dilemmas and problem situations, selection of tools and resources for teaching and barriers to students' learning (Zavlavsky, 2008). As part of a comprehensive study, which aimed to teach the concept of limit in a broad sense by focusing on the lesson planning, we focused to reveal how pre-service mathematics teachers reflect their mathematical knowledge in the concept of infinity in the context of the limit concept. The answer of this question provides important findings about the content of the method courses in pre-service teacher education, which usually includes preparing lesson plans.

Mathematics Teachers' Specialized Knowledge

In the teacher knowledge literature, there have been different models for examining mathematics teacher knowledge which have been based on Shulman's (1986) seminal work about teacher knowledge (Ball et al., 2008; Rowland et al., 2005). Considering the aim of the study, the knowledge that teachers might bring to classroom is of importance. For this reason, we adopted mathematics teachers' specialized knowledge (MTSK) (Carrillo-Yañez et al., 2018) as an analytical and methodological model which specifies mathematics teachers' knowledge to conduct their profession in not only teaching in classroom but also lesson planning or communicating with colleagues.

The two subdomains of MTSK (see **Figure 2**) are shaped by two common notions, which are about how students learn mathematics and how teachers should teach it (Carrillo-Yañez et al., 2018). Mathematical knowledge including mathematics content itself (knowledge of topics); the interlinking systems which bind the subject (knowledge of the structure of mathematics); and how one proceeds in mathematics (knowledge of practices in mathematics); in addition, pedagogical content knowledge including how mathematical content is taught in a powerful way (knowledge of mathematics teaching); how students learn mathematical content (knowledge of features of learning mathematics); and being aware of the curriculum specifications (knowledge of mathematics leaning standards).

Since the concept of infinity requires to be aware of it in a broader sense towards its nature in mathematical concepts, teachers' knowledge for the concept of infinity is included in different dimensions of knowledge models (Montes et al., 2014). The main focus of the lesson planning process was the concept of limit and we could observe the notion of infinity with the pre-service teachers' conceptions and their use of infinity to proceed mathematical knowledge related to limit concept. Moreover, lesson planning includes how to teach the concept as a result of the nature of the lesson planning process. Therefore, the focus of this investigation was given on KoT, KPM, and their relation with KMT which were described in detail below.

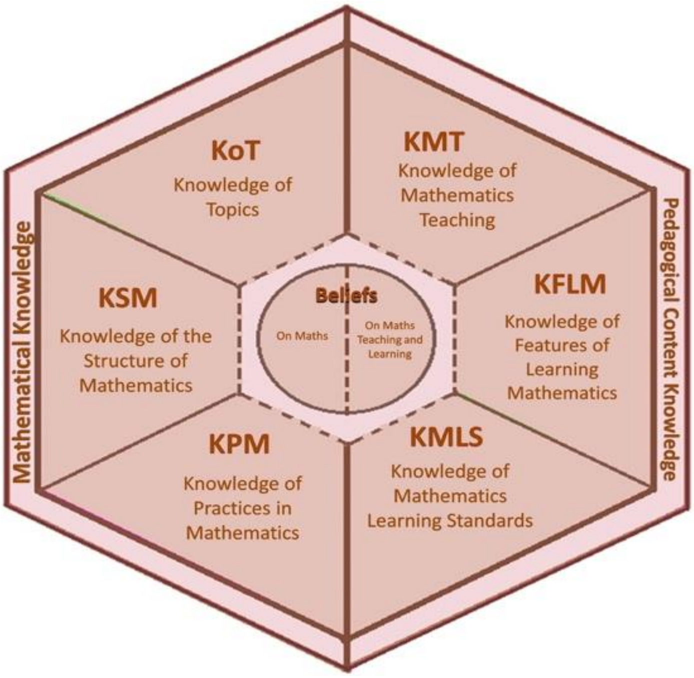

Figure 2. The model of MTSK (adapted from "the mathematics teachers' specialized knowledge (MTSK) model" by Carrillo-Yañez et al. (2018, p. 241).

It is possible to define KoT as the knowledge of concepts and the procedures related to a topic in mathematics curriculum as well as the insight on the conceptual basis for such concepts and procedures. KoT enables a pre-service teacher to characterize the concept in all its aspects, to read the mathematical meaning in a mathematical text, which enables students to infer deeper meanings (Carrillo-Yañez et al., 2018; Zakaryan & Ribeiro, 2019). It comprises knowledge of mathematical procedures, phenomenology and application, properties and foundations and representations. As described above, the notion of infinity constructs a foundation for the concept of limit for many sides. Therefore, we dealt with KoT in examining the pre-service teachers' knowledge in planning lessons for the concept of limit.

KPM, which is based on Schwab's (1978) syntactic knowledge, is a type of knowledge that encompasses ways of generating and advancing mathematical knowledge, verifying, how mathematics is constructed, mathematical reasoning, and exploring mathematics (Carrillo-Yañez et al., 2018). KPM enables mathematics teachers to establish connections between mathematical concepts to do mathematics. Planning a lesson about the concept of limit requires to be aware of how students do mathematics considering mathematical connections. The notion of infinity can be considered as an intersection of the concepts of calculus including limit, derivative and integral. Therefore, KPM can be observed as implicit mathematical knowledge which the lesson study group reveals during the planning process.

Infinity is of importance for professional knowledge which covers many related concepts in mathematical analysis (Montes et al., 2014). Since the curricula in many countries do not include infinity explicitly and the concept is not emphasized enough as teaching knowledge, what is not yet clear is how the mathematical knowledge of pre-service mathematics teachers reveal the notion of infinity in terms of KoT and KPM during the lesson planning process. As the literature suggests, lesson planning process is mainly based on knowledge of theories of mathematics teaching, teaching resources (physical and digital), and strategies, techniques, tasks and examples (KMT in this theoretical framework). In this way, the current study examined the mathematical knowledge of pre-service mathematics teachers in the concept of infinity in terms of KoT and KPM. Accordingly, the research questions are as follows:

1. How do pre-service mathematics teachers reveal their KoT and KPM regarding the concept of infinity in planning phases of the lesson study aiming to teach the concept of limit?
   a. How do the pre-service mathematics teachers' KoT and KPM relate with their KMT?

Table 1. The lesson plans and lesson study goals aimed during the lesson study process

Lesson plans	The lesson study goal of lesson plan
Lesson plan-1	Conceptualizing the fundamentals of the limit concept in students' mind
Lesson plan-2	Constructing the knowledge of the features of limit and of the applications within the context of limit except the ones whose result is infinity with the concept of limit

Table 2. The context of planning phases of lesson plans

	Phase 1	Phase 2
Lesson plan-1	• Discussion on how to conceptualize the limit concept in students' mind • Discussion on and preparing activities • If they do not have consensus on any proposed activity, revising or changing it	• Discussion on the reflections of the group on the effectiveness of lesson plan-1 in teaching a real class • Discussion on how to revise lesson plan-1 and preparing it for the next teaching • If necessary, changing the activities
Lesson plan-2	• Discussion on how to conceptualize the features of limit and infinity applications of limit in students' mind • Discussion on and preparing activities	• Discussion on the reflections of the group on the effectiveness of lesson plan-2 in teaching a real class • Discussion on how to revise lesson plan-2 and preparing it for the next teaching

METHOD

The current study employed the exploratory case study design which allows investigating an emerging phenomenon, which is not clearly specified and needs data for its construction, by means of insufficient preliminary theoretical assumptions in the research topic (Yin, 2014). This study provided us a way to in-depth explore pre-service mathematics teachers' KoT and KPM regarding the concept of infinity during their planning of lessons regarding the concept of limit.

Context and Participants

The study was conducted with a group of three pre-service mathematics teachers following the teaching practicum course in lesson study context. The lesson study process (see **Figure 1**) was designed by the researchers considering teaching of the concept of limit in a broad context. After the participants' consent to the study, there were two lesson study cycles conducted by the participants considering the lesson study goal that aimed to conceptualize the concept of limit in students' mind. The time of discussions in lesson planning was kept long enough to in-depth investigate their mathematical knowledge in a broader context. The participants planned two lesson plans based on the aims including fundamentals of and the applications with the limit concept (see **Table 1**).

Since the focus of this study was the planning phases of the lesson study process, the context of the study covered two lesson planning phases for each of the two lesson plans. As seen in **Table 2**, each lesson planning phase included discussion on the lesson study goals and preparing activities on these goals. Participants were provided readings for their needs before, during, and after planning processes.

Among non-probability sampling methods, the current study utilized purposive sampling method which is related to selecting participants based on some certain characteristics (Fraenkel et al., 2012). Since the lesson planning process requires sharing and exchanging knowledge to learn from each other, we looked for whether the participants had different perspectives and different levels of knowledge for teaching the concept of limit. In addition, in order to have a rich group discussion during the planning phase and to apply the lesson plans, which are a necessity of the lesson study process, in a real classroom environment, the participants were selected among the pre-service teachers who were in their last year of the four-year program of Mathematics Education. During the academic year in which the study was carried out, the three pre-service teachers (Mila, Fulya, and Alp, as their pseudonym) who volunteered to participate in the study were selected purposefully in terms of their experiences: Fulya and Alp had previously worked on the concept of limit and its teaching within the scope of a method course, while Mila did not have any experience on the subject of limit and infinity before.

Table 3. The numbers of lesson planning phases of lesson plans

Lesson plan	Phase	Number of meetings
Lesson plan-1	Phase 1	5
	Phase 2	4
Lesson plan-2	Phase 1	7
	Phase2	2
Lesson plan-3	Phase 1	2
	Phase 2	1

Table 4. The relation with the sub-domains of the model and the data gathered

The sub-domain	Its relation with infinity
KoT	The foundation of the concept
KPM	Ways of proceeding and communicating with the concept of infinity
KMT	How using the strategies, examples, analogies and metaphors

Data Collection Tools

The fundamental data collection techniques were the observation of the discussions in the planning phases of lesson study process, and the lesson plans and filed notes as relevant documents. There were 21 meetings for planning conducted during the lesson study process each of which took almost 2-3 hours (**Table 3**). The observation data were generated through video recordings and transcripts of the planning phases of the lesson study process. Video recordings were used not to miss any points in the planning. Also, there were the researchers' observation notes for each meeting of the planning phases.

It should be emphasized that the researchers are engaged with the lesson study group over a prolonged period that the first researcher has been the teaching assistant of them for three semesters in teaching methodology courses and the second researcher is the lecturer at the same university. Considering that the data collection methods have some limitations, the researchers sought to collect data based on different viewpoints and approaches to the research question. By this way, the reliability and validity were provided, which are important factors in qualitative studies.

Data Analysis

Before starting the data analysis, the researchers watched all the videos of planning lessons, matched the field notes and the transcriptions of all the video recordings of the planning phase of the lesson study process. Considering the aim of the study, we examined knowledge of infinity behind the activities of the participants. Since we had to give much attention for data analysis to reveal some implicit findings, we did it in two steps through utilisation of the pre-service teachers' ways of communication, and content analysis which offers analysis of people's behaviours (actions) (Fraenkel et al., 2012). First, the researchers separated the quotations of each participant referring to infinity or containing implicit knowledge of infinity. Second, the decomposed data were analysed concerning pre-determined concepts and themes (Corbin & Strauss, 2014) for which we used the indicators of KoT, KPM, and KMT as an organizational and methodological framework. **Table 4** showed the indicators for the concept of infinity for the sub-domains including KoT, KPM, and KMT. Codes were developed by examining excerpts of the lesson planning process and integrated into the indicators of sub-domains of MTSK by the researchers. It should be noted that KMT was not included in the codes at the beginning. The second research question occurred during the data analysis process to explain the data which were observed whether there was a relation between these sub-domains and KMT. We used the reliability calculation method of Miles et al. (2014) to analyse the coding of researchers comparatively. The inter-rater reliability was found to be about 80%. As the last step of coding, the researchers discussed the codes to reach a consensus.

Table 5. The two excerpts from the process of planning lesson plan-1

I. The first meeting for planning lesson plan-1	II. The third meeting for planning lesson plan-1
A: We can start with Zeno. Zeno's paradoxes. M: I don't think I know the paradoxes exactly, what are they? A: Well, in the paradox of Achilles and the tortoise, Achilles is in a footrace with the tortoise. Achilles allows the tortoise to start "x" meters ahead of him, for example. They are in such a race that Achilles never catches up with the tortoise, even though the tortoise is halfway through what Achilles takes. M: Interesting. Such a starting point would be effective to teach "approach".	F: I think that it is more useful to start the lesson with a problem. I designed a problem based on real life: "For the immunity game to be played in Survivor, one competitor from each team will shoot arrows from x meters away from the target. Regardless of the strength and skills of the players, each time they throw, they progress by half the current situation. Accordingly, which team wins the game? Why?" ... While preparing the activity, I thought that it may cause two misconceptions: One is "the limit value is never reached" and the other is "the limit is always equal to the value of the function at that point". M: Well...If they already know this paradox, they say neither of them can win, but if they don't, they definitely say both will win. I think there may be someone who says, we cannot know. A: I expect that students will answer this question that it will end equally. Since they already know the sequences and series, they may draw the formula for this function themselves and conclude that it will never reach.

Note. For this table and the rest of other tables, the acronyms A, M, and F represent that Alp (A), Mila (M), and Fulya (F)

FINDINGS

As mentioned above, there were two lesson study cycles conducted by the participants regarding the lesson study goals that aimed to conceptualize the concept of limit in students' mind in a broad sense. In this section, the findings are presented within excerpts based on the cycles of lesson study.

Lesson Plan-1: Fundamentals of Limit

In the beginning of the planning phase of the lesson study for the first lesson plan, the participants wanted to start with the aim of constructing the knowledge base for limit. In **Table 5**, there were two excerpts from the process of planning lesson plan-1. First, Alp proposed the idea of using one of Zeno's paradoxes, Achille and tortoise, as a step for underlying the idea of "approach".

The excerpt given in the first column is a part of the discussions on using the paradox. In the paradox Alp proposed, there is a never-ending process between Achilles and the tortoise. Since Alp demonstrated the paradox as an example for phenomenological aspects of infinity as potential infinity- "never-ending process", this excerpt shows that it is an example for KoT. In addition, Zeno's paradox can be considered as a dynamic representation of infinity. His KoT reflected his KMT as using examples, analogies and metaphors in techniques to teach the concept. In the latter section, it was shown that Alp used similar techniques when the topic is related to infinity. As opposed to Alp, Mila had some deficiencies about phenomenological aspects of infinity and making connections between infinity and limit, since she was surprised with this idea and she claimed that she had not known the paradoxes and the relation between paradoxes and limit.

The activity designed by Fulya with the use of a real-life problem was related to repeating process; halving the road repeatedly. The mathematical foundation of this activity is based on the concept of convergence of sequence. Similar with the previous activity the participants proposed, the process-oriented and iterative perspective of infinity (Montes & Carrillo, 2015, p. 3223) is regarded as potential infinity. For this reason, the excerpt can be considered in the domain of Fulya's KoT together with Alp. She reflected her KoT on KMT as a teaching strategy by using paradoxes in a problem.

Since both Achille cannot catch tortoise in the paradox and the arrow cannot reach the target board, the participants thought that it might cause a misconception in students' mind which implies that limit is an unreachable point. For this reason, they thought that the activity on Zeno's paradoxes couldn't serve the aim of the lesson plan. Before implementing the research lesson of the first lesson plan, participants changed their activity from Zeno's paradoxes to a task which included "approaching to 1" by filling the table (see **Figure 3**). This change can be explained as mini-cycling in the lesson planning process. In the discussion on this task, we observed an example for both KoT and KPM of participants.

The Activity: Let's say a number which is close to 1!

Activity-1: Conceptualize "approach" intuitively in students' minds

- Which of you can tell me the number closest to 1?
- Which of you can tell me the number farthest to 1?

! Expected answers: The student who says that the number 1.010 is closer than 1.01 is expected to divide these numbers into place values and compare them.
! Students are expected to understand the infinitesimal approach.
! Thinking of this as a game, students are asked whether this game will have winner.

Figure 3. The second activity the participants proposed

The aim of this activity was to conceptualize the notion of "approach" in students' minds. While preparing the activity, they proposed for the steps of the activity and the explanations for these steps as "Let's ask how these numbers behave (A)", "we will be approaching consecutive and endless moves (M)" and "Let's say if we approached infinitely small distances instead of endless moves (F)". It can be understood that Mila's expressions "get as close as" and "endless move" underlined the idea of infinitesimal. Like the previous example of Alp, such an expression which suggests a potential understanding of infinity (as the process is unfinished) can be considered as KoT for infinity. Fulya's suggestion as writing "approached infinitely small distances instead of endless moves" is related to ways of thinking (KPM). It can be interpreted as their KoT might affect their ways of thinking, KPM.

After they designed the lesson plan-1, one of the participants implemented it in a real classroom (it is called as "research lesson" in lesson study process) and the others observed her to collect data about the effectiveness of the lesson plan. By this way, they passed to the Phase-2 to revise their lesson plan according to their reflections about its effectiveness and whether it met the lesson goal.

Phase 2

In the research lesson of lesson plan-1, the activity did not work as effectively as they had expected with regard to students' learning and feedback. Therefore, they decided to change the activity for the beginning of the lesson plan. First, they focused on paradoxes again. However, in this time, we assigned them to read paradoxes different from Zeno's. By this way, they had a chance to reason the notion of infinity on paradoxes. Based on the video she watched while searching for infinity, Mila asserted a paradox that she called potato chips paradox. She indicated that the video should be shown to students to raise their awareness about the excellence of the notion. By this way, this attempt can be regarded as her KMT (showing video by teaching material). On the other hand, Alp proposed Hilbert's hotel paradox. **Table 6** shows a section of the discussion on paradoxes during lesson planning. In this process, Fulya just made comments on her friends' suggestions.

There was an important distinction between the claims of Mila and Alp in these excerpts. Alp thought on all these paradoxes and he reasoned on the difference of these paradoxes, rather than Mila. On the other hand, while Mila's asserted paradox showed an example for one-to-one correspondence, she did not reason on it about the difference between Zeno's one and this paradox. In addition, she did not realize that it represented actual infinity. While Alp's awareness can be considered as KPM, Mila's lack of awareness can be regarded as lack of KPM. On the other hand, Fulya went back to the Zeno's paradoxes. It can be understood that Fulya thought that the logic behind the paradoxes and accordingly the notion of infinity was considered as gradually approaching. Different from the first phase of planning, she tried to connect her conception which was potential infinity (KoT) related with one of the main questions of the planning "how do teach the concept of limit".

Table 6. Two sections (Mila and Alp) of the discussions on paradoxes during lesson planning

Mila	Alp	Fulya
M: I liked the video I made you watch about infinity: Look now! Let's imagine that there are infinite chips in an endless package of chips. One in 10 chips is burned. The neat ones and the burnt ones are matched. So, for each burnt chip, there is a regular chip that can be matched. What is happening? (Silence…) A: There was a hotel with infinite rooms similar to this. For example, you know there is a sequence of that chips' metaphor? F: Was it the alternative series? A: Wait, now I'm going to impress you. $+1-1+1-1$... It goes like this forever. If I took these, this place would be zero. If you take the binary like every time, it's zero. There is a unit here or my answer is 1. But if you take it like this, if you cut off this part and get zero, your answer comes to 0. Are you impressed? M: I was very impressed. I think the concept of infinity is a great thing! F: It works so well, it really is.	(When talking about the historical background of the limit in the first lesson plan) A: We can add it as a teacher reminder. Zeno had paradoxes, even in the last question, hotel-style things with rooms. We can write in 1-2 short sentences. For example; if the teacher is curious, he opens it and looks at the lesson plan because they are too long. So, writing the hotel with infinite rooms or something, writing the paradox of Zeno. F: Which of the paradoxes are we writing or why are we writing both? A: With Zeno's paradoxes, we'd better write a hotel with infinite rooms or something. We would indicate two different perspectives. Because they look at slightly different sides of infinity! M: How so? A: There is a never-ending process in Zeno's paradoxes, for example, the arrow never reaches the target board. But in the hotel paradox, if you've noticed there is a match. No matter how much the incoming guests increase, we can match hotel rooms with guests.	F: One of the problems was in the transition to formal definition. Maybe it could be like this: While we are giving Zeno's paradox, we will move to the coordinate system in step-by-step approach, or we can give the epsilon and delta when moving to the coordinate system. We also use it when we pass to the coordinate system when giving the epsilon delta. A: I don't understand. F: I mean, after the number line, we will switch to the coordinate system. M: I mean, we have to get to the coordinate plane somehow.

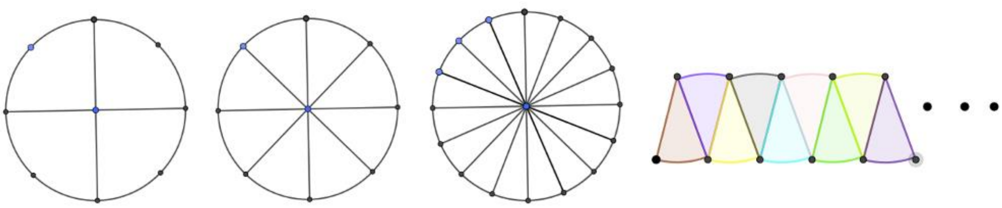

Figure 4. The demonstration of the revised activity

Among different ideas for the activity, they decided to make a connection between mathematical concepts including limit, geometry, and infinity in an activity named as "Finding the area of circle". In the activity, students are distributed circles in different sizes for each students' group in the classroom and the teacher wants them to cut or divide the circles into 4, 5, and 6 pieces. Then, they combine the pieces with one adjacent edge (see **Figure 4**) to obtain a parallelogram. Using the formula of area of parallelogram, they could reach the formula of area of circle. The following excerpt is a part of the re-planning phase of lesson plan-1.

> M: We will ask the area of the circle and the students will say πr^2, what if we ask how we can prove it, I mean how can we show it?
>
> …
>
> F: We will expect the student to use the limit here, right?
>
> A: Yes! They actually use the limit implicitly here. That third shape on the right here, as the area of the triangle in that shape increases, the thing looks like a parallelogram at first. Then, it approaches the rectangle. … It's a rectangle when you get 1 million or so sectors!

> …
>
> F: Actually, we get "Riemann sum" here, right? Starting from the infinitesimal calculus, we can actually take the limit of the sum of the series in infinity without saying integral!
>
> M: So actually, we can do it starting from this sector, just as we get a parallelogram when we divide the sectors into more and more smaller pieces which we cannot see clearly. The concept of limit is composed of small approximations in this way, Cauchy said so.

Though the excerpt given above doesn't include "infinity" explicitly, it shows pre-service teachers' knowledge about infinity. Mila's last interpretation about the activity, which includes "*more and more smaller pieces which we cannot see clearly*", indicates a significant property of the remaining item in a series, which is the underlying idea of the infinitesimal. The idea of infinitesimals directs us to potential infinity as phenomenological aspects of infinity. By this way, the interpretation of Mila is a clear example of KoT. Likewise, Alp demonstrated his KoT by his suggestion for the activity which is "*we may ask students how would it be if we did more than that?*". His question represents the idea of endless steps, since we don't know the meaning of "more than that". Otherwise, Fulya saw the mathematical idea behind the activity. Similar with Mila, she thought on "infinitesimal calculus". However, she focused on the result of addition of sectors' areas connected with Riemann sum. Since it can be said that she generated a mathematical way from the idea of infinity to limit and integral, it is one of the examples for KPM of Fulya.

Lesson Plan-2: Features and Applications of Limit

The second lesson plan aimed to construct the knowledge of the feature of limit of special functions including polynomial, trigonometric, exponential, logarithmic…etc. and to implement applications within the context of limit except in students' mind. Therefore, they focused on the concept of infinity more directly in Lesson Plan-2.

Phase-1

Before designing the lesson plan-2, the participants determined the topics in which they have faced difficulties throughout their own education life. Not surprisingly, they all focused on the topics related to the concept of infinity including limit at infinity, infinite limit, indeterminate and undefined forms. In the beginning of the designing the lesson plan-2, they started to discuss how they would refer the concept of infinity without confusing students. The researcher as a "*knowledgeable other*" directed group discussion on pre-service teachers' conceptions of infinity in order to deepen their discussions.

> R: What is on your mind? What does "infinity" mean to you? Let's say one by one.
>
> A: Like the 2001 space adventure movies. I mean, the space shuttle is going towards infinite black. It is like that. The infinity goes on like that (He shows it with his hands).
>
> F: I mean, the mathematical meaning comes to my mind directly. Well, there is a set which has a beginning but no end or there is a set which has an end, but no beginning. Like indetermined, so something indetermined.
>
> M: It is such an endless, far, far away … I mean, if we think of it as distance, it is a very, very far place. And, we don't know how far it is, but there is such a place, but we also know that. So, I think anything can happen there.

The answers of the participants did not surprise us, since their notions of potential infinity were observed in planning phase of lesson plan-1. First, Alp presented his idea about the infinity which was a mysterious thing like a scene in the space movie. His metaphor is similar with his example-Zeno's paradox-given in lesson plan-1. It can be said that his knowledge for teaching on infinity is shaped by his conception of infinity. Mila, as well, thought that infinity is an endless distance which cannot be measured. As mentioned above, unsurprisingly, this shows us their KoT in terms of their conceptualization of potential infinity. On the other hand, Fulya approached the question in a different way which was related to the concept of "set". It did not

Table 7. The excerpts about limits at infinity or infinite limit

Mila	Fulya
M: The thing here is that the limit is actually infinite and it gives us the asymptote. The asymptote is actually something related to the limit, so this means that I am moving on a curve and I have a line. They are gradually getting closer to each other at infinity and the distance between them is approaching, so it happens again through approaching.	(…) F: In the meantime, you have asked why, more precisely, what would happen if the limit was infinite? I read it from the Calculus book because infinity is not a number, so we cannot talk about the limit when going to infinity, but we think that the function behaves towards infinity. By this way, we talk about the infinite limit. (…) It provides a way of describing the way of function that grow arbitrarily large positive or negative. … M: It actually allows us to see the behavior. F: Actually, we cannot say there is a limit, but it allows us to see the behavior.

mean that Fulya referred either set theory or actual infinity. Rather, it meant that Fulya might have an awareness towards the actual infinity (KoT).

There were several important issues considering the notion of infinity in lesson plan-2 including limits at infinity, infinite limit and mathematical procedures with infinity as we mentioned in the introduction. The participants built the lesson plan-2 on these notions given above. **Table 7** showed some examples about limits at infinity or infinite limit from the planning process.

First, it should be mentioned that the explanations of the participants could be considered as right for infinite limit and limit at infinity. In these examples, we focused on their ways of thinking (KPM) about how they handled the notion of infinity in these concepts. Mila's explanation covered both limit at infinity and infinite limit. The expression to describe infinity in her explanation that included "getting closer" shows us her ways of thinking towards the potential meaning of infinity. Like Mila, Fulya's ways of thinking towards the potential meaning of infinity included the expression as "grow arbitrarily large". However, Fulya mentioned that infinity is not a number insistently. It could be explained as Fulya considered infinity as an entity which is different from a number.

Phase-2

After conducting the research lesson of the lesson plan-2, the participants tried to revise the activities, problems or exercises that did not work in the research lesson. For instance, they discussed on the concept of infinity again based on Fulya's (as a teacher of the lesson) expressions for infinity as a number. This was really interesting that she had mentioned that infinity is not a number insistently in Phase 1. Based on this claim, they focused on the idea of how they should express the concept of infinity. By questioning each other's knowledge during the discussion in the revision process in planning, the pre-service teachers had a chance to make sense of their knowledge. In the following excerpt, Alp asked his friends whether there can be limited infinity in mathematics. Such a question triggered other participants to think both on their knowledge and Alp's knowledge.

> F: I think we can talk about the infinity as something which is "constantly increasing".
>
> M: Yes, I read about that! Infinity is not a quantity; it is a quality. Then, it may be sensible! It says we use the concept of infinity as an adjective in mathematics. We do not use it as a noun, the article says, like a finite adjective, infinite is an adjective used in mathematics. This means that infinite is the opposite of finite. In other words, things that are not finite in mathematics are called infinite.
>
> A: Well, could there be bounded infinity?
>
> M: What do you mean?
>
> A: Constantly increasing cannot be considered as wrong. However, what about bounded infinity? If we say bounded infinity, for example, there is a bounded infinity between 0 and 1. However, there are infinite numbers in this interval.

M: Yes, there could be.

A: However, it is so close. I mean that the place between 1 and 0 is too short.

M: But, close according to who?

A: It seems this much close to me (showing that there was a very short distance using his thumb and index finger).

M: Too far for me. For example, this distance may be too close for you, but it may be too far for me. I mean, it depends. So, we can think of it as a quality from this perspective. I remember that there is a one-to-one correspondence in sets. I think this issue is related to it.

The excerpt showed Alp's mathematical knowledge for infinity in two sub-domains. First, when Alp asked his friends what about "*bounded infinity*" by relating it, it showed us that he used infinity as a mathematical object. Though he didn't indicate Cantor's one-to-one correspondence explicitly, he referred to it by being aware of the existence of infinity. Bearing this situation in mind, the excerpt showed both his KoT and KPM. On the other hand, the excerpt showed the effectiveness of the discussions of lesson planning. Even if only one member of the group has this information, he/she can enable other group members to gain a different perspective on this issue during the discussions in lesson planning. This excerpt did not show the evidence for the reflection KoT and/or KPM on KMT.

DISCUSSION AND CONCLUSIONS

The current study investigated the pre-service mathematics teachers' mathematical knowledge about infinity within the lens of KoT and KPM and in the model of MTSK in the lesson planning phases aiming to teach the concept of limit in lesson study process. As a part of a longitudinal study, the lesson planning which was organized to enable the pre-service mathematics teachers to have deep discussions on the concept and multi-directional knowledge development. The findings showed that they explained the concept of infinity as a gradual approach to the limit concept, underlining the potential perspective considering the fact that the concept of limit is a phenomenological aspect of infinity, by referring to Zeno's paradoxes of Achille and the tortoise, and later Dichotomy. Generally, the same finding was observed in KPM of the participants as ways of thinking the notion of infinity in planning to teach the concept of limit. As Kidron and Tall (2015) indicated, it can be considered as an expected result in terms of the participants' tendency to potential infinity, since they were likely to consider the concept of limit as an infinite process which is never-ending and never-reaching a goal (Date-Huxtable et al., 2018; Monaghan, 2001; Montes & Carillo, 2015). However, they thought and discussed on more than just the potential meanings of infinity, such as a never-ending process or something unlimited.

As Umugiraneza et al. (2018) indicated that lesson planning process requires thinking on the aspects of teaching from multiple perspectives including learning activities, knowledge for teaching the related topic, resources, assessment strategies, it provided a way to participants about thinking on and enhancing their mathematical knowledge by means of discussing how to teach the concept. When we compared the first and second cycle of lesson planning process, it can be observed clearly that participants had some changes in their KoT and KPM in the concept of infinity. For instance, while Alp focused more on the potential meaning of infinity with both the metaphors he used and the paradoxes he suggested for lesson plans, he was able to look at the concept of infinity from a modern perspective with the idea of limited infinity in the second cycle of lesson planning. There are several possible explanations for these results. Though the current study showed only lesson planning process of a longitudinal research, the participants had a chance to conduct their lesson plans in a real classroom between these two lesson planning processes. This chance might make them think on a broader perspective to mathematical knowledge with students' perspective beyond what they think and see. Another possible explanation could be the discussions and time to think on these discussions during the planning process. As mentioned in the method section, the planning process took longer than similar studies to provide space for participants for developing their knowledge. One of the most effective ways for teacher learning is to collaborate with colleagues for thinking on teaching and solving problems related to student learning. Social interaction between teachers in collaboration has shown that teachers support each other's knowledge building (Rock & Wilson, 2005). The current study supported this claim with the second excerpt in the lesson planning phase 2 in which Alp asserted different perspectives for both KoT, KPM, and KMT about the actual meaning of infinity. Alp's directive questions in the discussions made participants to think about the notion of infinity from different perspectives. However, there is also the other side of the coin. Participants' use of each other's knowledge may, in some cases, cause the participant with lack of knowledge to be affected by the knowledge of the others, such as Mila in this study, to stay within that border. In the case of Mila, it can be said that Mila constructed her own KoT within the lines of the other participants' KoT reflected in planning.

While the first phase of lesson planning did not affect any other factors different from the pre-service teachers' existing mathematical knowledge, the second phase of lesson planning comprised both planning lessons and reflecting of the enactment of the planned lessons. Therefore, the effect of reflective stance should be considered in the change from the first and second phases of planning. Similarly, the findings showed that the observation of the planned lessons brought them different perspectives about the notion of infinity in teaching limit. In the literature, the reflection in the planning phase is used for developing the pre-service mathematics teachers' knowledge of students and content (KFLM for this theoretical framework) (Shuilleabhain, 2016). When considered from this point of view, the answer of what the role of reflection in planning lesson for development can be explained as standing in students' shoes.

One of these different perspectives about the notion of infinity they brought after the first lesson plan conducted was to discuss the complexity of infinity. At this stage, the question arises of what has changed so that they have begun to argue about the complexity of infinity. The findings can explain this fact as the iterations in lesson planning. As stated by Zavlavsky (2008), knowledge construction could occur in an iterative process which requires thinking about it and making necessary corrections as knowledge development is achieved, conducting lesson plans provided participants to think on the concept in a knowledge construction process. In the current study, the iteration between lesson planning might stimulate them to think more on the notion of infinity for effective teaching of the limit concept. In other words, we can consider the development of pre-service teachers in this process as a snowball growth. They might have had the opportunity to look from a different perspective in the second cycle, with the development of the knowledge they did not have or lacked in the first cycle.

In future studies, other sub-domains of mathematical knowledge and pedagogical content knowledge may also be included in the study. In addition, the concept of infinity should not be considered as a concept only for secondary level students. Rather, the concept is equally important for primary and elementary school students and teachers (Bozkus et al., 2015; Date-Huxtable et al., 2018; Jirotková & Littler, 2004). For this reason, analyzing the knowledge for teaching the concept of infinity mathematics of primary and elementary school teachers and pre-service teachers about infinity or related elements can bring a different perspective to the literature.

Considering that students have difficulty in visualizing and learning, they will have cleverly prepared questions in the classroom for the concept of infinity which provide a basis for lots of mathematical concepts. The well-educated mathematics teachers are of importance to make reasonable explanations when students ask smart questions or when they ask anything related to course that may have valuable contribution to their learning even if it is not worded decently. For this reason, the results of the study can be extended to construct a progression for development of knowledge of infinity for mathematics teachers at secondary and elementary schools and also primary school teachers.

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