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Mathematical induction: deductive logic perspective

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Abstract
Many studies mentioned the deductive nature of Mathematical Induction (MI) proofs but almost all fell short in explaining its potential role in the formation of the misconceptions reported in the literature. This paper is the first of its kind looking at the misconceptions from the perspective of the absence of the deductive logic from one's conceptual scheme of Mathematical Induction proofs. In light of this framework, we give a detailed description of the Tower of Hanoi game as a model that can be used to emphasize the Deductive Reasoning in MI proofs.

Keywords: Mathematical Induction, Recursion, Proof Techniques, Deductive Reasoning, Mathematical Modeling.

Introduction

Not only novice learners but also some practitioners and professionals sometimes express a sense of "*not a real proof, just a bunch of algebraic manipulations*" referring to the proof by Mathematical Induction (MI). Indeed, to too many novice learners, Mathematical Induction proof is shown to be a challenging topic pedagogically, epistemologically, and cognitively. Even after formally seeing a Mathematical Induction (MI) proof, many college students still struggle with its meaning. Existing studies report that many of these students resist the use of Inductive Hypothesis without considering its absolute truth value. Consequently, the same group is not able to cope with how a statement to be proven may become assumed [*true*] in the structure of the MI proof itself (Avital and Libeskind, 1978; Ernest, 1984; Fischbein and Engel, 1989; Harel, 2002; Harel & Sowder, 1998; Ron and Dreyfus, 2004).

Mathematical Induction proof is among the three categories of the documented proof schemes: external conviction, empirical and deductive proof (Harel & Sowder, 1998). In fact, it is an essential deductive proof technique especially for the verification of the truth state of infinite recursive sequences. The concept of recurrence lies at the heart of Mathematical Induction proofs (Ernest, 1984). It is the key feature that allows one to consider the conditional proposition, $P(n) \Rightarrow P(n+1)$, as a "*variable inference,…,a place holder for the entire sequence of inferences*" (Harel, 2002).Thus, if a learner does not recognize the necessity of the inherent recursion in the deductive proof of the Inductive Step's true value then he/she may not be able to grasp Mathematical Induction as a valid proof technique.

The Mathematical Induction Principle (MIP) stated below is at the core of the ideas employed by MI proofs (Avital & Libeskind, 1978; Ernest, 1984):

Mathematical Induction Principle (MIP):

Let P(n) be a mathematical proposition corresponding to a natural number, n.

If:

i. ***Basis****: For a natural number k, P(k) is true;*

ii. ***Induction Hypothesis****: Assume P(n) foreach $n \geq k$.*

*iii. **Inductive Step**: The conditional statement* $[P(n) \Rightarrow P(n+1)]$ *is true fo rall n ≥ k;*
Then,
P(n) is true for all natural numbers n ≥ k.

The Mathematical Induction Principle (MIP) is stated symbolically as well:
$(P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]]) \Rightarrow \forall n \geq kP(n)$

Deductive Logic in MI Proofs

We believe deductive reasoning is what makes it possible one to arrive at the truth state of infinitely many propositions without verifying the truth state of each individually. The particular type of reasoning occurs first when considering the truth value of the Inductive Step, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$, and next when making an inference about the truth value of $\forall n \geq kP(n)$.

Considering that MIP is a conditional proposition, let us first analyze the deductive logicof a generic proposition $P \Rightarrow Q$ via its truth values, given in Table 1. The truth value of a conditional statement is determined by the truth values of its components. As seen on the table, the conditional statement is true when P and Q are both either true, or both false. Alternatively, the proposition $P \Rightarrow Q$ is true regardless of the truth value of P as long as Q is true.

Table 1. Truth values for $P \Rightarrow Q$

Cases	P	Q	$P \Rightarrow Q$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

In fact, the proof by Mathematical Induction employs the first case shown in table 1. That is, MI proofs deduce the true truth value of the consequence of MIP via Modus Ponens. In other words, if the conditional statement is accepted and the antecedent (P) holds, then the consequent (Q) is inferred. Thus, in light of the truth value of MIP (a conditional proposition),and the inferred true state of the Inductive Step, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$ coupled with the concretely verified true value of the Basis Step, *P(k)*,MI proofs infer the true value of the consequence, $\forall n \geq kP(n)$.

Specifically, in MIP, the antecedent is $P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$, and the consequence is $\forall n \geq kP(n)$. Considering that MI proofs employ Modus Ponens, for a valid inference of the truth state of the consequence, one needs only to verify the truth value of the proposition, $P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$.

Misconceptions

In the literature, misconceptions about MI proofs is covered mainly under the following three topics:

1) Informal Logic
2) Inductive Reasoning
3) Deductive Nature of MI Proofs

Informal Logic

Informal rules of logic appear to alter one's ability to accurately look at the mathematical logic of an argument. Baker (1996) for example reported his high school students' reliance on informal logic in their proof by Mathematical Induction. These students seemed to attribute the truth state of a mathematical statement to the accuracy of a proof. Some of his participants, to give an example,considered that an incorrect proof would imply a false mathematical statement. Another group of participants on the other hand believed that the falsehood of a statement would imply a wrong proof.

Another area of MI proofs adversely affected by the informal logic is its Induction Hypothesis component. This component is applied incorrectly by many learners due to the commonly used meaning of the term "*Assume.*" The term is used in daily language mainly in place of "*Consider True.*" Thus, many students, functioning with the informal logic, take "*Assume P(n)*" as being "*Consider that P(n) is true.*" Consequently, many feel uncomfortable with the feeling of "*Already considering P(n)as true, yet proving its truth value again.*"This line of logic makes many of the novice learners form strong doubts about the validity of MI proofs. The following excerpt clearly reveals Ernest's participants' use of the daily meaning of "*Assume,*" and their consequential distrusts:"*why do we need to prove P(n)'s truth if we already assumed the truth of P(n)?*" Ernest (1984).

Preference for the informal logic rules may be explained by the ideas of Fuzzy Logic discussed in Zazkis (1995). According to Zazkis, "*…fuzzy thinking appears to be more intuitive, simple, and preferred by many students.*" She adds furthermore that "*the lack of symmetry between the antonyms true and false is one of the major pitfalls in students' understanding of formal logic*" (Zazkis, 1995). Therefore, dictated by fuzzy logic, learners may be overgeneralizing the statement, $\forall n \geq kP(n)$, thus ignoring the formal logic of the particular proposition being true only when each proposition, *P(n) for* $n \geq k$, is true (Ernest, 1984).

Inductive Reasoning

Another source of misconceptions is the confusion of "*Inductive Reasoning*" with the term "*Induction*" used in the statement of MI proofs. It has been shown that in the traditional instructions, the most dominant proof schemes are authoritative proof schemes focusing mainly on the pattern generalization, which is, according to Harel and Sowder (1998), a manifestation of the empirical proof scheme. Thus, such mathematical instructions may be reinforcing the learners' interpretation of the term "*Induction*" as an empirical method for which to arrive at a conjecture based on a few computed values.

Contrary to the inductive reasoning, deductive reasoning calls for making inferences about the behavior of a task based on the collective properties among its components. It is a logical process in which a conclusion drawn from a set of assumptions that contain no more information than the premises taken collectively. This behavior mirrors the variable inference form of the Inductive Step in MI proofs (Harel, 2002). That is, at the particular step, inferences are considered as placeholders for the entire sequence of inferences to be drawn so that there would be no need to run specific inference steps such as: $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$ and so on (Harel, 2002).

Under the influence of inductive reasoning, many learners attempt to verify the truth of each, $P(1)$, $P(2), P(3)$ and so on, one by one at least for the first few of them before conjecturing on the truth of all,*P(n)*. Ernest (1984) for instance reported that many of his students attempted to apply Inductive Reasoning to Mathematical Induction proofs. He adds that this resulted in his students arguing the truth value of infinite cases on the basis of the truth of a few specific cases. Apparently, this in turn caused some of his participants desire to verify one-by-one all of the earlier cases before making a decision on the truth value of a later cases. Precisely, in an infinite recursive sequence, to prove the true value of the 11th term, many students in Ernest (1984) wanted to first verify the true truth value of all the terms up-to and including the 10th statement.

Teachers of mathematics are no exception to such misconceptions (Ron & Dreyfus, 2004; Stylianides et el. 2007). Ron and Dreyfus (2004) gave one teacher's description as to what he did with his students about the Inductive Basis, and Inductive Step in a proof by Mathematical Induction lesson: *"one student went home and checked all the cases up to 10 rings and basing on this we succeeded to prove that it is true also for 11 rings?"* This excerpt clearly indicates that the particular teacher not only was advocating Inductive Reasoning but also holding an understanding of MI proof as an empirical proof scheme.

Deductive Nature of MI Proofs

Many studies mentioned the deductive nature of MI proofs but almost all fell short in explaining the potential role of the *deductive logic* in the formation of the misconceptions reported in the literature. This paper is the first of its kind looking at the misconceptions from the perspective of the absence of deductive logic from one's conceptual scheme. We discuss the following misconceptions from a view point of the deductive logic:

1) Circular Appearance of Inductive Step, andLimited Validity ofInduction Hypothesis.
2) Unnecessity of Basis Step.
3) MI Proofs as Algebraic Manipulations
4) Unawareness of Recursion's role.

Circular Appearance of Inductive Step, and Limited Validity of Induction Hypothesis

Deductive Logic of $\forall(n \geq k)[P(n) \Rightarrow P(n+1)$

In mathematical induction proofs, deductive logic occurs in making inferences about the truth value of the inductive step, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)$. Inductive Step has to be proven independently as a self-regulating statement with no regards to the truth state of the earlier statements. In other words, at this step, learners need to be able to work with the generic proposition, *P(n),* without considering its absolute truth value.

As we discussed earlier, the statement "*Assume P(n) for an n ≥ k,"* included in mathematics textbooks as the Induction Hypothesis step, seems to confuse many students to why they assume *P(n)*[*meaning to be true*],yet prove its truth again. Evidently, these lines of reasoning give rise to the feelings of one making unnecessary circular arguments. Fischbein and Engel's (1989) findings for example indicate that the Induction Hypothesis was considered as an isolated step to be shown, but not as the integral component of the Inductive Step. According to Fischbein and Engel, only 28% of the students in their study gave responses that reflect a conceptual understanding of both the Induction Hypothesis and its deductive role in the Inductive Step. Apparently many of their students tended to consult the absolute truth value of the Induction Hypothesis, *P(n),*for a specific value of n, in verifying the truth value of the Inductive Step, $P(n) \Rightarrow P(n+1)$, even though in deductive processes, such truth value is irrelevant. The other 72% of their students distinctly shows preference for empirical processes.

Fischbein and Engel (1989) discussed a group of participants assigning a limited validity to the Induction Hypothesis. The particular group seemed to believe in the possibility that for certain circumstances, the Induction Hypothesis may not hold. Assignment of a limited validity to the Induction Hypothesis is highly likely the manifestation of a mental scheme holding more of inductive reasoning structures than that of deductive logic ideas.

It is obvious that both groups lacked an understanding of the Inductive Step as a deductive step rather both displayed behaviors supporting empirical processes. If you recall, deductive logic dictates that the conditional, $P(n) \Rightarrow P(n+1)$, is proven without assigning any truth value for each proposition, $P(n)$, $P(n+1)$, individually, and without regards to the numerical value of n. Empirical processes on the other hand seek for the concrete verification of the proposition, *P(n),*individually for each numerical value of n.

According to Ernest (1984), tendencies to assign absolute truth value to Induction Hypothesis led his participants too forming a similar circular view of "*the method in which we assume what you have to prove, and then prove it*," and that "*it has a suspicious likeness to assuming what you have to prove!"*Baker's (1996) college student participants also revealed a similar view of Mathematical Induction. Many in his study made arguments tangent to the statement: "*you assume what you want to prove*." Fischbein and Engel reasoned that their students made such statements most likely due to uneasiness with the acceptance of: "*the entire segment of the induction step on a statement which itself has not been proven, and cannot be proven in this segment of the reasoning process*" (Fischbein and Engel, 1989). We ascribe their participants' uneasiness, with the acceptance of proving a statement based on an unproven proposition, to the lack of deductive reasoning from their conceptual scheme of the Inductive Step. That is, a learner with an understanding of MI proofs that does not integrate the deductive logic is naturally expected to feel uncomfortable working with statements that have not yet been verified concretely.

Unfortunately, the teachers of mathematics display tangible misconceptions about the Induction Hypothesis and the Inductive Step. For instance, Ron and Dreyfus (2004) documented some of their teachers considering the Induction Hypothesis with a limited validity. One of their teachers stated during an interview that:

> *"…we go to the hypothesis…if it works why do we have to assume that it works up to k? because I say, If I see that it works then i assume that it will work until a certain stage. But, I can't assume that it will work forever because it can stop working like all the electric tools do…my task is to prove that it will also work in the next stage" (p. 119).*

We construe the phrase, "*I see that it works,… my task is to prove that it will also work in the next stage,"* as, at the Inductive Step, this teacher wanting to first verify concretely the true state of the proposition, *P(n),*for a specific value n, and then verify the true value of the next proposition, *P(n+1)*.This teacher undoubtedly displays an empirical process understanding of MI proofs, thus, at the time, s/he may not have been aware of the deductive nature of the Inductive Step. Indeed, his notion of empirical process is clearly seen on the excerpt: "*I can't assume that it will work forever because it can stop working like all the electric tools do."* In other words, s/he did not appear to verify the true truth value of, " $P(n) \Rightarrow P(n+1)$," as a single proposition without regards to the truth state of the proposition, *P(n),* or the numerical value of n. On the contrary, this teacher shows tendency to check each proposition one by one. To justify his reasoning for checking individually, he says: "*because it can stop working*."We believe analogies such as "*electrical tools*" may further reinforce reasoning with empirical processes. Mathematics teachers introducing the Inductive Step with such analogies may be instilling MI proofs as Inductive processes. Then, one may need models that focus more on the deductive processes in order for learners to begin functioning with deductive logic at the Inductive Step. Hence, later in the paper, we discuss Tower of Hanoi as one such model.

Deductive Logic of $\forall n \geq kP(n)$

We believe some participants reported in the literature (Baker, 1996; Ernest, 1984; Fischbein and Engel, 1989) formed a circular notion of MI proofs due to their inability to distinguish the truth state of the proposition, $\forall n \geq kP(n)$, from the truth states of the statement, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$. For instance, according to the 4th case in table 1, the conditional proposition, $[P(n) \Rightarrow P(n+1)]$, gets a true truth value as long as the statements *P(n)* and *P(n+1)* are both false for some n-values, whereas the only time the proposition, $\forall n \geq kP(n)$, gets a true value is when the statement, *P(n),* is true for all n-values. For example, consider the following simple quantified conditional proposition, $\forall n > 2 \in N, [(n = 2) \Rightarrow (n+1) = 3]$. In fact, this proposition is true even though both its components are false. Thus, not being aware of the deductive logic differences between the two propositions of the type, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$, and $\forall n \geq kP(n)$, learners may inaccurately conclude that the statement, $\forall n > 2 \in N, n = 2$, is true due to the true value of the conditional statement, $\forall n > 2 \in N, [(n = 2) \Rightarrow (n+1) = 3]$.

Many learners as a matter of fact appear to consider (applying the 1st case in table 1) the conditional $[P(n) \Rightarrow P(n+1)]$ having true value only when both propositions *P(n)* and *P(n+1)* are true. Thus, not being able to recognize the various logic cases for the Inductive Step, these learners consider the Inductive Step and the Conclusion component of MIP identical. This then results in them instinctively thinking they already proved the true value of *P(n)* by proving the true state of the conditional $[P(n) \Rightarrow P(n+1)]$. Accordingly, they feel there is no reason to prove the truth value of *P(n)* again.

Unnecessity of Basis Step

Another misconception reported in the literature is about the role of Basis Step in MI proofs. Indeed, studies report that many students even some instructors appear to consider the Basis Step as an independent step to carry out without regards to its deductive implication to the components of MIP. Consequently, failing to recognize the role of the Basis Step, many learners either rely on the specific examples in their arguments (Ernest, 1984; Fischbein and Engel, 1989) or skipped the Basis Step entirely (Baker, 1996; Ernest, 1984).

According to Ron and Dreyfus (2004), teachers of mathematics are no exception to forming these types of misconceptions. Failing to recognize the crucial role Basis step plays in the deductive inference of the truth of the Conclusion component, $\forall n \geq kP(n)$, many teachers in their study considered the action of verifying Basis Step as a motivation for the proof. Ron and Dreyfus (2004) stated that one of their teacher participants for instance used an electrical tool as an analogy to tell his/her students that "*I want to buy a used tool. Before I bargain about the price, I plug it into see if it is worth bargaining*." Here, one can see that the particular teacher is clearly using an irrelevant aspect of the model in order to make a case for the necessity of the Basis Step. This teacher seems to use the tool analogy as an action that justifies the proving effort, but not as an integral part of the deductive process employed by MI proofs. Novice learners with a similar notion of basis step may for instance verify the basis step for *k*=2 as a justification for starting the MI proof, and arrive at the incorrect deduction for the truth of the conditional, $\forall n \geq 2 \in N, n = 2$.

Indeed, this kind of behavior is due to the lack of deductive reasoning in one's understanding of the MIP statement:

$$[P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]] \Rightarrow \forall n \geq kP(n).$$

In a deductive process by means of modus ponens (employing the 1st case in table 1), one needs to prove the truth value of the statement $P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$ in order to be able to make an accurate inference about the truth of the conclusion component of MIP, $\forall n \geq kP(n)$. The truth value of the logical statement $P(k) \wedge \forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$ depends both on the truth value of the Basis case, *P(k),* and on the truth value of the Inductive Step, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$. In fact, in MI proofs, the true value of the statement, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]$ is necessary but not sufficient to infer the true truth value of the proposition, $\forall n \geq kP(n)$ .As a matter of fact, focusing only on the truth state of the inductive step may result in the inaccurate deductions of the MIP conclusion.

For instance, ignoring the Basis Step, in cases where the Inductive Step is true when each *P(n)* and *P(n+1)* is false, learner may inaccurately infer the true state of $\forall n \geq kP(n)$. In these cases, Basis Step plays a gate keeper role that prevents erroneous conclusions. Here is another scenario in which incorrect deductions may be made. Consider the cases where *P(n)* and *P(n+1)* are true for all but a few earlier cases, in these cases, the role of the basis step is to identify the very first n-value where *P(n)* is true. Again, failing to recognize the earlier cases where *P(n)* is false, learners may mistakenly conclude its true value for all natural numbers.

MI Proofs as Algebraic Manipulations

Failing to accurately grasp the role of deductive reasoning in MI proofs, many novice learners turn their attention to the structure and symbolism over substance. This in turn leads to the conceptualization of MI proofs to be only a set of procedures to carry out. Baker (1996) for instance gives the following excerpts as a testament to some of his students' procedural understanding: "*Yes, it started with an equation and proved it true through a number of different steps or assumptions*." Ernest's participants were not exempt from such conceptualizations either. Some of his participants stated that: "*when using the method in free variable form we assume the inductive hypothesis* [meaning the true value of *P(n)*] *go through a* [meaningless] *complicated procedures and end up having proved* [*P(n)*]" (Ernest, 1984).

Unfortunately, the type of problems used in the introduction of MI proofs further adds fuel to the fire. Mathematical Induction proof presents itself in problems many of which can be addressed via algebraic manipulations, thus reinforcing students' notion of MI proofs being a bunch of meaningless manipulations. For instance, when college students are asked to prove that the statement, " $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \forall n \in N$ " is true, most consider this task to be the algebraic manipulation of simply adding the term "*n+1*" to both sides of the equation (Harel, 2002). As a matter of fact, such tasks lack the necessary characteristics to provide the learning environments that are effective in bringing the deductive process into view. As long as learners obtain acceptable responses (confirmed by an external authority) via algebraic manipulations, many feel satisfied with their responses, and as a result go on without studying the role of deductive logic in the conceptual framework of MI proofs.

Functioning only with procedural knowledge, learner may furthermore inaccurately infer the true state of $\forall n \geq kP(n)$. Let's consider our example from above, $\forall n \geq 2 \in N, [(n=2) \Rightarrow (n+1)=3]$. We already discussed the fact that this quantified conditional is true for any value of *n*. Working with only the tools of procedural knowledge, learners may simply verify the truth of *P(2)*, and obtain $(n+1)=3$ from n=2 via a meaningless algebraic manipulation of adding 1 to both sides of the equation, n=2. As a result, they may come to the erroneous conclusion of the true truth value for $\forall n \geq 2 \in N, n=2$.

According to Trigueros and Ursini, (2003), novice learners tend to attribute a fixed meaning to a symbol especially when they function only with their procedural knowledge. This behavior is also evident in MI cases. In fact, the procedural understanding appears to result in the confusions about the role of symbolic representations used in the MI proofs. Ernest (1984) for example reports his participants' irrelevant understanding of the notations included in MIP. He indeed talks about the confusions his students displayed when the symbol "*n*" is used both in the inductive step, $\forall (n \geq k)[P(n) \Rightarrow P(n+1)]$, and in the conclusion step, $\forall n \geq kP(n)$. In his study, dictated by their procedural knowledge, many participants appeared to have assigned a single meaning to the symbol "*n*," and as a result get confused on why they assumed the truth of the proposition, *P(n)*,for *n* at the inductive step, and at the end, proved the proposition *P(n)* for the same *n*-value. To them, *n* seems to be referring to the same fixed value at both inductive and concluding steps even though MI proofs call for two different meanings for the symbol *n.* MI proofs indeed consider *n* at the Inductive step representing a generic constant, and at the conclusion step, a variable that can take any value greater or equal to *k* (Trigueros & Ursini, 2003).

Additionally, as mentioned earlier, this line of reasoning may lead to the formation of MI proofs containing unnecessary circular processes. Similarly, it may also become the validating factor for the learners' interpretation of "*assume P(n)*" being "*take P(n) true*" if the symbol "*n*" is considered as having one meaning in both at the inductive and concluding steps.

Unawareness of Recursion's Role

Ignorance of the deductive reasoning required at the Inductive Step is further elevated by the unawareness of the role inherent recursion plays at this step. To be exact, in the absence of the absolute truth of each of the propositions, *P(n)* and *P(n+1)*,without knowing the specific value of *n*, only the inherent recursion makes it possible the deductive verification of the truth value of $[P(n) \Rightarrow P(n+1)]$ .In other words, by obtaining *P(n+1)* from *P*(*n*), recursion shows the equivalence of the propositions, *P(n)* and *P(n+1)*,at their truth states (i.e. both are either true or false),thus deductively proves the true state of the conditional $[P(n) \Rightarrow P(n+1)]$. It does this regardless of the value of *n*. For example, consider the following conditional proposition, "$[(n=2)] \Rightarrow [(n+1)=3]$ for a natural number *n*." It is not possible to prove this conditional based on the truth value of each component since, in this example, the specific value of *n* is not known, thus, the truth state of its components cannot be determined. For that reason, it can only be proven deductively via the inherent recursion by obtaining n+1=3 from n=2, that is, adding 1 to both sides of the equation, n=2.

Many learners treat the task-inherent recursion as if it is only a part of the algebraic manipulations, and consequently do not put much attention to its crucial role in the deductive verifications. For instance in the finite task case, "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \forall n \in N$," adding "*n+1*" to both sides is considered, by many students, as an algebraic manipulation rather than as a recursion that makes possible the proof of the true truth value of the conditional, $[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \Rightarrow (\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}]$, without knowing whether each component, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$ is true or false. Thus, many lack an understanding of the absolute necessity of recursions in the deductive confirmation of the true value of conditional statements, $[P(n) \Rightarrow P(n+1)]$. Hence, many may fail to recognize that in tasks with no recursive behavior, the conditional at the inductive step cannot be proven; hence MIP would not be a valid option.

Ron and Dreyfus (2004) indeed relate some of their students' difficulties to the lack of understanding of the recursion's role at the Inductive Step. Consequently, they recommend to give students opportunities to gain confidence with recursion and its function through a naive approach, namely to show how the truth of the statement for *n=2* follows from its truth for *n=1*, the truth for *n=3* from the truth for *n=2*, and so on. They further add that this can be done through models with problem solving situations.

> "*A role of models can be to demonstrate, illustrate and interpret the method of proof by mathematical induction, and thus to support understanding by pictorial language that might be more accessible to learners than the formal language commonly used in teaching mathematical induction*" (Ron and Dreyfus, 2004; p. 114).

Dubinsky argues that students will continue to be unsuccessful with Mathematical Induction proofs as long as teaching methodologies continue to ignore the cognitive obstacles that students experience. There are a few recommended instructional approaches reported in the literature. One such approach is that of Dubinsky's (1989) computer-based teaching method following a genetic decomposition of Mathematical Induction proofs. For more details of the decomposition, see both Dubinsky (1989), and Dubinsky and Lewin (1986). Apparently, his computer-based instruction provides students means (via programing) to construct an object formation of the propositions included in the Mathematical Induction Principle. According to Dubinsky (1989), in order to document the effect of the particular approach, some of his students were given computer-based activities. Each activity targeted one component of the genetic decomposition of MI proofs. He adds that once students completed the activities, many of the participants were able to form a deeper understanding of MI proofs, and moreover they were able to recognize the variety of situations calling for a proof by Mathematical Induction, and apply it correctly.

Model Comparison

Domino model is the most commonly used model in demonstrating the role of recursion in MI Proofs. What sets Domino model apart from the computer-based decomposition of MI proofs is its visual characteristics. Domino model in fact focuses more on the visual aspects of the inherent recursion. Its visual properties furthermore are easily accessible. At times, however, this fact may lead to a quick decision with the satisfaction of one's knowledge regardless of the relevance. See table 2 below for a summary of the features of Domino Model. Contrary to the computer-based programing approach, Domino model lacks any relevance to the mathematical tasks on hand. Thus, many learners, without an actual problem to guide, are not able to weed out the irrelevant aspects from the relevant ones hence end up forming an understanding of MI proofs mainly based on the irrelevant features. According to Ron and Dreyfus (2004), many teachers in their study for instance judged the recursion as meaning that all the dominos are with equal distances yet close enough to each other, completely missing the recursion's role in the deductive proof of the inductive step.

Domino model in fact is used right up at the beginning to support the definition of MI proofs with no connection to any realistic problem situation, failing to provide a meaningful experience. Consequently, Inductive Step stays an artificial step to be carried out to too many learners.

To reiterate, we believe the Domino model without a relevant meaningful problem solving situation may still make the use of MIP looks, to a novice learner, like an artificial process. As a matter of fact, we as the instructors of college mathematics, observed many of our college students totally disregard the inherent recursion behavior despite the use of Domino Model in our lectures. Therefore, we strongly deem the integration of relevant models that are not easily manipulated. We believe one needs not just a model that provides geometric means, but needs a learning environment that effectively put into view the crucial role recursion plays at the Inductive Step. Tower Hanoi model is one such model that can elicit a strong awareness.

Here, we give a short comparison of the domino model with the Tower Hanoi problem. See table 2 below for an outline. Later in the paper, we will also provide a detail discussion of the Tower of Hanoi model, and its relevance to the components of MI proofs.

Similar to Domino Model, Tower of Hanoi too provides the visual representations of the inherent recursion behavior. More importantly though this model, contrary to Domino Model, is structured within a realistic problem situation providing more meaningful and relevant experiences. As a result, it decreases the likelihood of making irrelevant interpretations. That is, if a visual characteristics noticed by a learner do not fit into the problem situation, it is likely to be deemed as irrelevant hence disregarded rather than embraced. In other words, the problem situation guides learners toward the relevant features of the analogy hence diminishing the possibility of irrelevant interpretations.

In addition, Tower of Hanoi model is complex enough that it does not give easy access to the irrelevant features of its visual representations, as a result decreases the possibility of one forming an understanding exclusively based on the extraneous characteristics.

Table 2. Model Comparison. Domino vs. Tower of Hanoi

Domino Model	**Tower of Hanoi**
No relevance to a mathematical task Awareness of recursion only at the visual level; artificial meaningless knowledge Gives a sense of unnecessary work	In a context of a relevant problem situation providing meaningful experience Awareness of recursion both visually and algebraically; knowledge becomes meaningful with the problem given Reinforces necessity of components of MI proof
Easy access to visual characteristics leading to quick decision of satisfaction with knowledge Breed incorrect interpretations	Provide visual mode Highly complex, no easy access to irrelevant features thus eliminating possible irrelevant interpretations
Sense of irrelevance thus understanding of MI as artificial process breeding procedural knowledge	Relevance to an actual problem to solve leading to a relevant conceptual understanding.
Completely ignore deductive role of recursion in the algebraic proof of Inductive step.	Higher difficulty level of computational approaches in addressing the problem reinforces the need and the necessity of recursion in the algebraic proof of Inductive step.

Tower of Hanoi Model

Tower of Hanoi provides a mathematical task to be solved (find its statement below), which can elicit an effective and accurate interpretation of the components of MI proofs. Similar to other models, this model too provides visual modes (see fig.1 below). But, its visual modes are complex enough that it leaves very little room for irrelevant interpretations. To be exact, even though learners are able to easily conjecture a pattern solely playing the game for small quantities, it becomes almost impossible to visually verify the same pattern for larger values. Consequently, one has to revert to an approach that eliminates the need for the repeated application of the recursive pattern. It is the algebraic approach (via deductive consideration of MIP) that diminishes the need for repeated computations.

This algebraic process moreover brings out the need for the application of the recursive behavior in a deductive process.

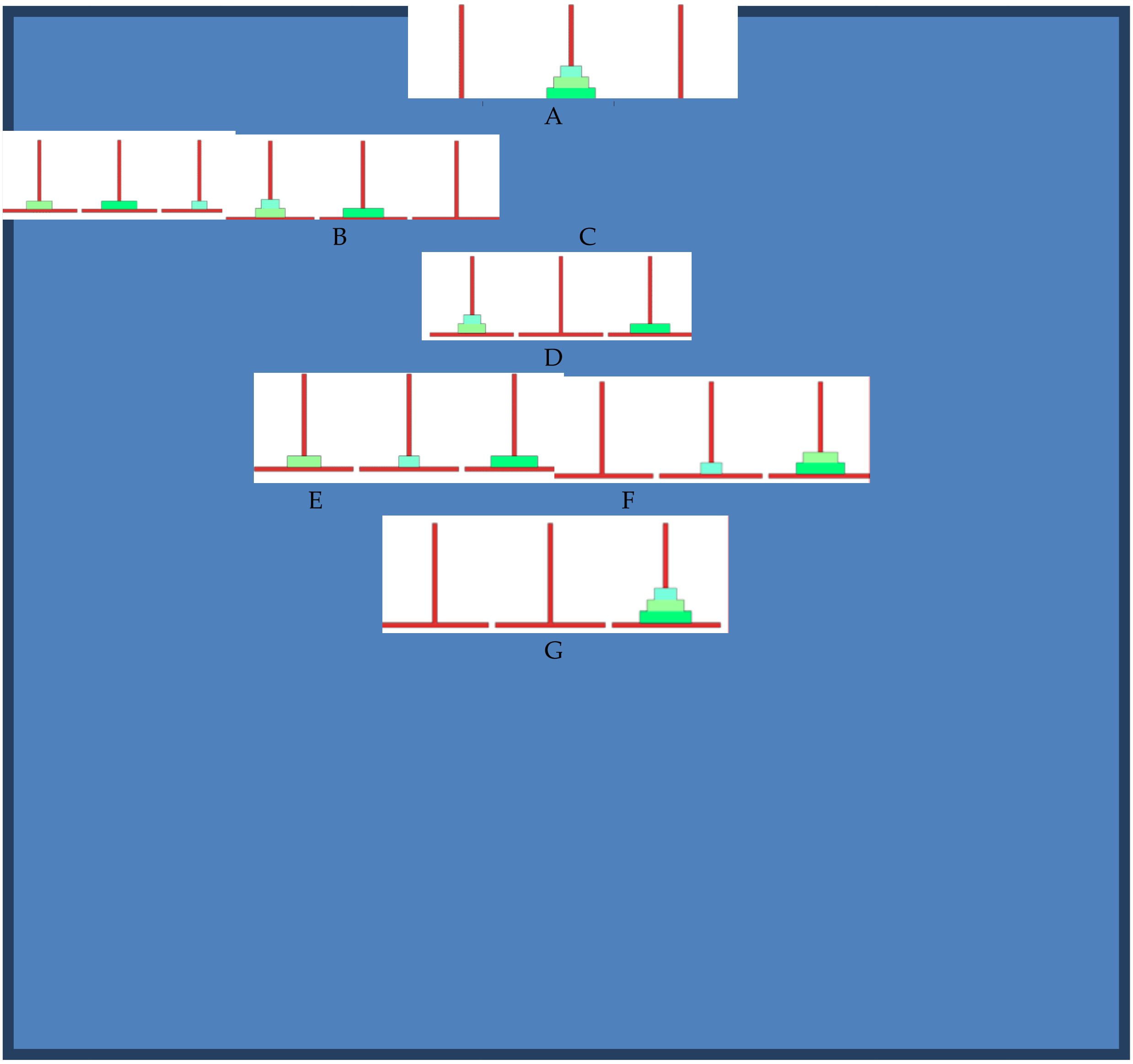

Figure 1.Illustration of the inherent recursion via moving three disks from the second tower (A) tothe third tower (G). First two disks are moved to the first tower (A-C). Next, the last disk (larger one) moved to the third tower (C-D). Finally, repeating the earlier moves (A-C), two smaller disks located on the first tower (D) moved to the last tower (D-G).

> ***Tower of Hanoi Problem:***
> *Consider the towers and disks shown in Figure 1 below. What is the minimum number of moves one needs in order to move 1000 disks from one tower to another following the two rules: (1) Only one disk can be moved at a time; (2) No disk may be placed on top of a smaller disk.*

Recall that models like Dominos make it easy to visualize the recursive behavior even for higher quantities, thus giving a false sense of satisfaction. But, many of them fail to provoke any need for other approaches. Even though Tower of Hanoi is similar to the Domino model in the sense that it also provides pictorial representations (see Fig. 1), it is drastically different in the complexity of conceptualizing the recursive pattern for a larger quantity of disks. See Table 3 below for an outline of the provoked behaviors due to the Tower of Hanoi activity.

Recursion

Tower of Hanoi Model is one of the suitable models for the visual introduction of recursion activities, but also is effective in provoking learners to think about the recursion's crucial function in proving the Inductive Step.

The recursive behavior in the game comes from the fact that for the transfer of (k+1) disks from one tower to the next, firstly k-many disks are moved from one tower to the next, secondly, the remaining disk, $(k+1)^{th}$ disk, is moved to a separate tower (see fig. 1, frames A to D). Furthermore, the same moves used in the earlier step to transfer the first *k-disks* are repeated to move the very disks back onto the present location of the $(k+1)^{th}$ disk (Fig. 1, frames D to G) .As seen in the description above, an algebraic recursive behavior emerges naturally from the game itself. That is, the minimum number of moves needed to move *(k+1)* many disks (call it M_{k+1}) is double the minimum number of moves needed to move *k* many disks (M_k), plus one. Then, the recursion formula inherent in the game is given algebraically: $M_{k+1}=2M_k+1$. It can easily be confirmed that the act of moving disks indeed leaves no room for learners to direct their attention to the irrelevant features of the disks or the towers. In fact, the problem on hand guides players to focus on the moves at all times. Another trait of the model is clearly present in the description above. That is, contrary to many models with visualization modes, Tower Hanoi not only provides means to observe visually the inherent recursion but also provides needs (solving a mathematical problem) to translate this recursion into an algebraic expression.

As an example, Figure 1 above illustrates, for three disks, the recursive nature of the Tower of Hanoi game. On the frame A, you see the three disks located on the second tower. The next frames, B to C, depicts the transfer of the first two disks to the first tower keeping in mind the condition of making minimum number of moves (using 3 moves). The frame D illustrates the relocation of the last disk (third disk; the largest size) from the second tower to the third tower (using a single move). Finally, the frames, E to G, demonstrate the act of repeating the steps, used earlier for the first two disks (frames B to C), in moving the very same two disks back onto the third tower (again using 3 moves). Note that just now you in fact experienced the visual verification of the minimum moves via the repeated application of the recursion. This recursion can also be translated into an algebraic formula: $M_3= 2 M_2+1=7$where$M_2=3$ so that the question on hand can be answered computationally.

Once students discover, and gain confidence with the inherent recursive pattern in the Tower of Hanoi game, playing with a small quantity of disks such as 3 or 4 disks, instructors may the nask the question of how to determine the minimum number of moves for 1000 disks (see the statement of the problem above).It should be noted here that computing the moves with the repeated application of the recursive formula even for 100 disks would be a daunting task not only for a human, but even for a computational technology.

After several attempts, recognizing the high difficulty level of their arithmetic approach of applying recursion repeatedly, it is highly plausible that many of them become discouraged from applying computations further. Thus leading to a search for more effective approaches to address the problem. This quality indeed lacks from the models like Dominos. Since there is no problem to solve with such models, the urge to seek for other more effective means does not materialize.

Out of the need to eliminate the repeated arithmetic application of the recursion, learners may begin searching for another algebraic expression (a closed formula) that does not require a recursion in its arithmetic evaluation. As a result, they may be able to discover the non-recursive formula: $M_n=2^n-1$.If however learners fail to arrive at the particular formula on their own, it can be given to them only after going through a set of guided investigative activities. Next step then is to have learners think about the validity of this formula for the larger number of disks. In fact, the verification of this expression can be achieved by applying the recursion from the first disk to second and so on, but the execution of this process is almost impossible for larger quantities. This in turn may lead to the realization of the necessity of a mathematical proof that does not call for the verification of each step with repeated computations. It is then, out of the need, the "*Proof by Mathematical Induction,*" emerges naturally as a deductive proof technique. Again, models similar to Dominos lack such features that can elicit the need for deductive proof techniques.

Inductive Step

Realizing that the repeated recursion application is no longer an option, learners may be guided to think about the recursion's help in the deductive verification of the Inductive Step. At this point, since many students already aware of the impossibility of proving the formula for larger n-values with repeated arithmetic, they may be more inclined to work with the Inductive Step without assigning an absolute truth value to either statements, $M_n=2^n-1$ *and* $M_{k+1}=2^{n+1}-1$. Hence they may begin to consider the role recursion plays in the proof of, $[P(n) \Rightarrow P(n+1)]$, as a whole. Here, we consider *P(n)* standing for $M_n=2^n-1$.

That is to say, in order to prove the conditional statement, $[P(n) \Rightarrow P(n+1)]$, one has to verify that the proposition, *P(n+1)* (namely $M_{k+1}=2^{n+1}-1$) is obtained from the proposition, *P(n)* (namely $M_n=2^n-1$), without knowing whether the statement, $M_n=2^n-1$ is true or false for any n. Learners then may come to the conclusion that the proof of $[P(n) \Rightarrow P(n+1)]$ can only be achieved deductively with the application of the recursive formula, $M_{n+1}=2\ M_n+1$. That is, applying the recursion, we arrive at $M_{n+1}=2\ (2^n-1)+1=\ 2^{n+1}-1$, replacing 2^n-1 with M_n. Note that with this argument, learners don't need to consider the truth state of $M_n=2^n-1$. That is, at this stage, even in the case of the falsehood of the proposition *P(n)*, the conditional $[P(n) \Rightarrow P(n+1)]$ could be true.

In short, it is clear that Tower of Hanoi leaves very little room for learners to misinterpret the visual recursion inherent in the game. Furthermore it is almost impossible for learners to address the problem without the algebraic application of the recursion, especially for the larger numbers of disks. Therefore, the model leaves no room for a possibility of learners ignoring the recursion. Indeed, the game coerces one to become fully aware of the role recursion plays in answering the question, otherwise cannot be addressed.

Role of Basis Step

To expand the activity, one can ask learners if/whether the true value of $[(M_n = 2^n - 1) \Rightarrow (M_{n+1} = 2^{n+1} - 1)]$ (that they just proved deductively applying the recursion) would be sufficient enough to make accurate inferences about the true state of the proposition, *P(n)=:* $(M_n = 2^n - 1)$, individually for each n-value. In turn, this may bring out the role of the Basis Step in MI proofs.

At this point, learners' attention may be directed to the truth values given in Table 1 above, especially the 4th case. They can be pointed to the fact that it is possible to deductively obtain a true value for $[P(k) \Rightarrow P(k+1)]$ while each of its components is false. To reinforce student exposure to similar truth cases, they can be asked to consider the following simple proposition: "$[(n = 2)] \Rightarrow [(n+1) = 3]$ *for n.*" As discussed earlier, this proposition is always true, yet for many n-values neither of two statements, $[(n = 2)], [(n+1) = 3]$, is true. Having seen examples of the 4th case in table 1, learners then may be guided to question the condition(s) needed to guarantee the true value for each *P(n)* individually. Students next can be directed to consider Modus Ponens on MIP.

That is, they may be asked to apply (to MIP) the first truth case in table 1 above to deduce the truth state of the conclusion, $\forall n(M_n = 2^n - 1)$ .Since the deduction of truth of $\forall n(M_n = 2^n - 1)$, depends on the differing truth states of the two MIP components, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$ and $\forall n \geq kP(n)$, comparisons should then put forward discussions on the logic combinations, $[T \Rightarrow T]$, $[F \Rightarrow F]$ and $[F \Rightarrow T]$.

Considering that these three truth combinations deductively infer the true value for the conditionals of the form, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$, students may begin a process of self-questioning, in their quest to determine whether these truth combinations also result in the true state of the conclusion, $\forall n \geq kP(n)$ of MIP. As a result, they may realize that at least for the two truth combinations, $[F \Rightarrow F]$ and $[F \Rightarrow T]$, despite the true value of the inductive proposition $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$, the quantified statement $\forall n \geq kP(n)$ is false. That is, learners may recognize that the falsehood of the antecedent implies the existence of at least one *n-value* (greater than or equal to *k)* where *P(n)* is false.

While applying Modus Ponens to MIP, students then may become aware of the impossibility of a valid deduction of the truth of $\forall n \geq kP(n)$,solely based on the true state of the inductive step, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$. Thus, some may begin to ask what other condition(s) is needed so that the valid inference of the truth state of the conclusion $\forall n \geq kP(n)$ becomes plausible. Already proven the Inductive portion of MIP, $\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$, learners may then turn their attention to the Basis Step component $[P(k)]$ (Basis Step). In turn, they may become aware of the necessity (in the Modus Ponens application on MIP) of the Basis Step for a valid inference of the true truth value of the formula, $M_n = 2^n - 1$, individually for each n-value.

Notice that addressing both recursion and the basis step in the context of the Tower of Hanoi game not only diminishes the possibility of irrelevant interpretation of the visual features, it also eliminates the formation of a feeling of recursion being only a procedural tool for meaningless algebraic manipulations. Most importantly, it coerces learners into thinking about the role deductive logic plays in MI proofs, moreover the role of Basis Step in this deductive process. See Table 3 for an outline.

Table 3. Outline of Tower of Hanoi Activity

Activity	**Provoked Behavior**
Play the game for smaller number of disks	Discovery of the visual inherent recursion and Need for an algebraic recursive formula: $M_{n+1}=2M_n+1$
Attempt to calculate M_{1000} using recursive formula	Awareness of the shortcomings of the repeated application of the recursive arithmetic, and the need for closed formula: $M_n=2^n-1$.
Attempt to verify the validity of $M_n=2^n-1$ for large n=1000	Realization of computational difficulty, and elicit the need for a deductive validation approach (MI proof).
Attempt to Prove the Inductive Step $[P(n) \Rightarrow P(n+1)]$	Impossibility of computational verification of each P(n), and need to prove the conditional without knowing the truth values hence realization of o need for the application of the inherent recursion.
Truth Analysis of the components of MIP: Comparison of similarities and differences of	Realization of need for the true state of P(k) (basis step) for a valid inference of the true value

$\forall(n \geq k)[P(n) \Rightarrow P(n+1)]]$ and $\forall n \geq kP(n)$ of conclusion, $\forall n \geq kP(n)$.
via the truth combinations: $[T \Rightarrow T]$, $[F \Rightarrow T]$
and $[F \Rightarrow F]$.

In summary, the expectation is that, through both internal and external questioning/guidance within the Tower of Hanoi problem, learners may come to the realization that:

1) MI proofs are the application of Modus Ponens on MIP, needing deductive reasoning.
2) *Induction Hypothesis* does not mean assuming the true truth value of *P(n)*individually. In fact, at this stage, *P(n)* cannot be assumed to be neither true nor false.

3) At the *Inductive Step*, the true truth value of the conditional statement, $[P(n) \Rightarrow P(n+1)]$ is proven as a whole for any n-value, without regards to the absolute truth of *P(n)*.More importantly, this can be done only with the application of the inherent recursion in task on hand.
4) In light of the true value of the proposition, $[P(n) \Rightarrow P(n+1)]$, and the true state of MIP ,Modus Ponens dictates that the only time the quantified proposition $\forall n \geq kP(n)$ can be inferred to be true if the Basis Step, *P(k),* is also true for ak-value.

Conclusion

We reported the pedagogical and cognitive difficulties students experience with Mathematical Induction. Furthermore, in light of the recommendations reported in literature, we talked about the approaches and models used in introducing MI proofs. We added new perspective into the existing one by discussing these issues in the context of the deductive logic applied in MI proofs.

We came to a conclusion that the most appropriate approaches to addressing these obstacles need to be through mathematical models with problem solving situations with relatively high difficulty level of computations. Due to the hardship of the computations for larger quantities, we argued that such problems may effectively elicit the cognitive awareness of the role and the necessity of each component of Mathematical Induction proofs. We additionally argued that the models with relevant problem settings diminish the likelihood of irrelevant interpretations. On the contrary, introducing Mathematical Induction proofs with topics such as finite sums, we deemed, lead to the formation of procedural knowledge, and furthermore to a view of MI proofs as unnecessary circular arguments.

In conclusion, for an effective instruction of MI proofs, we recommend the instructors of mathematics to discuss the recursive nature of sequences specifically in the context of deductive reasoning. We further recommend that learners are discouraged from concrete computations by directing their attention to the larger values. Instead, learners should be guided toward the discovery of a non-recursive closed formula, in turn the questioning of the validity of the closed formula for larger quantities may bring out the need for MI proofs, and the role deductive processes play in the proof of its components.

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