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Research Article

Is there room for conjectures in mathematics? The role of dynamic geometry environments

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ABSTRACT

Proof, as a central and integral part of mathematics, is an essential component of mathematical education and is considered as the basic procedure for revealing the truth of mathematical propositions and for teaching productive reasoning as part of human civilization. Is there therefore room for conjectures in mathematics? In this paper after discussing at a theoretical level the concepts of proof and conjecture, both in a paper-and-pencil environment and in a dynamic geometry environment (DGE) as well as how school practice affects them, we fully explain a task involving various mathematical disciplines, which we tackle using elementary mathematics, in a mathematics education context. On the occasion of the Greek educational system we refer to some parameters of the teaching of geometry in school and we propose an activity, within a DGE, that could enable students to be guided in the formulation and exploration of conjectures. Finally, we discuss the teaching implications of this activity and make some suggestions.

Keywords: conjecture, proof, geometric transformations, dynamic geometry environments, mathematics education

INTRODUCTION

When information and communication technologies (ICTs) are utilized in mathematics education, attention should be paid to the choice of activities that will hold up the development of mathematical understanding as well as to the tools that may be useful to students (McCulloch et al., 2018; Sherman, 2014). In particular when dynamic geometry software (Cabri Geometry, GeoGebra, The Geometer's Sketchpad, etc.) are used in the classroom as virtual laboratories in which students can explore and learn mathematics (Arcavi & Hadas, 2000), mathematics teacher is given the opportunity to organize interactive activities that integrate digital tools to generate mathematical reasoning and understanding of geometric proofs, two fundamental aspects of mathematics (Jeannotte & Kieran, 2017; NCTM, 2000, p. 56).

Various new possibilities have opened up for the teaching process with the emergence of dynamic mathematical programs, updated facilities available in computer algebraic systems and provided graphical tools in geometric environments (Velichová, 2021). Besides, the students who were taught mathematics with the use of technology showed a higher level of conceptual understanding compared to the students who learned using the traditional method (Liburd & Jen, 2021).

Nowadays it is widely accepted that teaching must keep pace with the development of technology (Winter et al., 2021; Zhao & Liu, 2019). So then, mathematics teacher, especially in geometry, has to include in his/her lesson, activities such that engage students, provide them with appropriate tools for posing and testing conjectures, and make it easier to formulate mathematical reasoning and compose rigorous proofs.

However, everyday school practice does not always follow the above context and many exercises and tasks are approached procedurally rather than conceptually (Crooks & Alibali, 2014; Rittle-Johnson et al., 2015; Star & Stylianides, 2013). At the same time, the teaching of geometry tends to be arithmetized, i.e., it is taught as an axiomatic presentation, which presupposes the system of real numbers (Patronis & Thomaidis, 1997). For example, the following exercise is included (with numerous variations) in many geometry textbooks all over the world, but it is essentially an algebra exercise: "In an orthogonal triangle, one angle is equal to 2/3 of another angle. Calculate all the angles of the triangle".

Particularly in the Greek educational system with which we are intimately familiar, the teaching of geometry is limited to the comparison of triangles, geometric calculations, and basic elements of analytic geometry (vectors and lines in the plane), while of the plane loci, only the circle, the bisector and the perpendicular bisector appear occasionally in some paragraphs of the textbooks (see Rizos & Adam, 2022). Also, both the geometric constructions with straightedge and compass in a paper-and-pencil environment and the use of dynamic geometry software are extremely restricted and essentially rely on the initiatives and skills of each teacher. Thus, with few opportunities for experimentation and conjecture, students are mostly focused on solving techniques of theoretical and calculation exercises.

In this paper, in order to highlight some of the challenges facing the teaching of mathematics in secondary education, we first deal with an exercise from the official Algebra textbook of the 11th grade of the Greek high school. Then, with the help of digital technology, we make a conjecture as an extension of the exercise, a conjecture, which we validate, arriving at a rather unexpected finding. Finally, we discuss the teaching implications of this activity and make some suggestions. This case study is particularly unique to the Greek educational system, and we assume that it could be useful to other educational systems around the world. We attempt to integrate the concept of conjecture alongside the process of mathematical proof in daily school practice, as well as the more appropriate use of dynamic geometry environments (DGEs) in the context of mathematics education. By studying the current curriculum, we may be able to improve certain elements of the conventional learning process and enhance students' mathematical understanding by merging different approaches into one task.

A good problem is not finished when it is solved. Its solution only rekindles more curiosity about related or peripheral ideas (Wares, 2018, p. 156). That is, perceptual curiosity combined with creative thinking often transcends activities designed for one level and merges into the study of more advanced ideas at a higher cognitive level (Abramovich et al., 2019). When confronting an issue, students should be encouraged to create their own reasoning and arguments. Moving from argumentation as a process (i.e., production of arguments) to argumentation as a product (i.e., expression of arguments) is not taken for granted (Albano & Dello Iacono, 2019). The nature of arguments, which students refer to not only depends on the culture of theorems developed in the classroom, but also strongly relies on the nature of the task; by their very nature, some tasks induce children to produce and/or exploit empirical arguments (measurements, visual evidence, etc.) (Boero, 1999). We believe that the task we chose to deal with satisfies the above standards.

THEORETICAL ISSUES

There is a spectrum of multiple learning theories and educational models, each concentrating on a different aspect and a different process inside the classroom environment. Bruner (1961) assumed that the learning process is not about the transmission of knowledge by the teacher, but about facilitating students' thinking and felt that learners generate new ideas or conceptions based on knowledge through this process. Learners select and convert information, form hypotheses and give their decisions based the structure of the process of awareness (Tran et al., 2014). This is the concept behind Bruner's (1961) discovery learning. In DGE, discovery learning can be achieved by the tools provided by the applications (for example, "drag" and "drop", "intersect", "show trace", and more). The action of dragging can allow the user to "feel" motion dependency, which can be interpreted in terms of logical dependency (Mariotti, 2014). Conjectures generated through maintaining dragging seem to come with a strong theoretical rupture between their premise and their conclusion, due to the lack of theoretical evidence found during the process (Baccaglini-Frank, 2019, p. 789). Furinghetti and Paola (2003) think that only through experimenting personally the construction of parts of a theory (under the guidance of the teacher and in situations carefully projected) students may give up, when necessary, the perceptive level and appreciate the meaning of theories. The teacher needs to be present and guide the students into contributing to the task themselves and explore into mathematical thinking, nor to dictate the process neither stay absent and only observe.

To make students to construct parts of a theory means to allow them to experience the construction of mathematical knowledge at different levels: the level of exploring within particular cases, those of observing regularities, of producing conjectures, of validating them inside theories (which may be already constructed or in progress). In developing this approach we are concerned with the transition from elementary to advanced mathematical thinking (Furinghetti & Paola, 2003, p. 398). But there is a deep connection of the students' mindsets with the formal proof of a hypotheses or a conjecture. The certainty the traditional method of learning provides seems important for them. Although processes of conjecture-generation may be facilitated by the dynamic component (whether they are given by a software or only by the mind of the conjecturer), proofs need to be constructed within the theory, which in traditional mathematics is "static". Passing from the phenomenological world to the mathematical world through the production of conditional statements is not trivial (Baccaglini-Frank & Mariotti, 2010, p. 227). Data suggest that the price is an incompleteness of theoretical evidence with respect to what is needed to prove the conjecture (Baccaglini-frank, 2019, p. 790). Modern teaching should focus on the process of introducing the students to conjecture making and verifying via the use of modern dynamic geometry platforms and the insight caused by visualization of a task. In our case, such technologies are utilized in order to find the locus for the conjecture. The concept of investigating the locus using DGEs has been seen in similar research (Oxman & Stupel, 2022; Segal et al., 2015). Through that perspective, there can theoretically a unique, unusual, and distinct orbit of the educational process be proposed.

In the literature about mathematics, there is a formal establishment of the concept of proof, and it is indicated that the proving process is an integral part of the curriculum in terms of emphasizing the relationships between the learning areas (Yilmaz Akkurt & Durmus, 2022). There are some specific stages in approaching a problem and getting to the proof. The stages consist of understanding the problem, exploring the problem, formulating a conjecture, justifying the conjecture, and proving the conjecture (Astawa et al., 2018, p.17-18). This means that conjecture making is needed as a prior step in order to get to the formal proof. Students do not have an a priori need for rigor proof, but they naturally make conjectures based in observation depending on the task. The traditional sequence "definition→theorem→proof→exercise" does not seem to have didactic success neither historical basis (Rizos & Adam, 2022, p. 3). In mathematical school activities as well as in mathematics curriculum, both formal and informal modes of proof and argument must be combined (Boero, 1999; Hanna, 2020). So, the stage of proof and the stage of conjecture making are both essential for the teaching process in mathematics education and mathematics teachers have to provide their students with all the necessary mathematical tools to enable them to both make and test conjectures and to do rigorous proofs.

Even though 11th grade school textbooks (as much as some textbooks of the previous grades) contain references and specific paragraphs about transformations on the two-dimensional plane and relevant activities that could strengthen those references, never such issues are included in the curriculum or even are discussed in the school practice.

THE PROBLEM AND ITS EXTENSIONS

The problem we have chosen to deal with comes from the official and unique Algebra textbook of the 11th grade of the Greek secondary education and has been part of the curriculum for the last thirty years. The problem is the following: Find the intersection point of the following lines, according to the value of the real parameter *α*:

$$l_1: ax + y = a^2$$
$$l_2: x + ay = 1$$

Once high school students have been taught that an equation of the form $Ax + By + C = 0$, $A \neq 0$ or $B \neq 0$ represents a straight line and know that the above linear system is parametric, it is expected that someone will solve the problem using the method of determinants. Then, we follow the standard steps in order to solve the problem based on school knowledge.

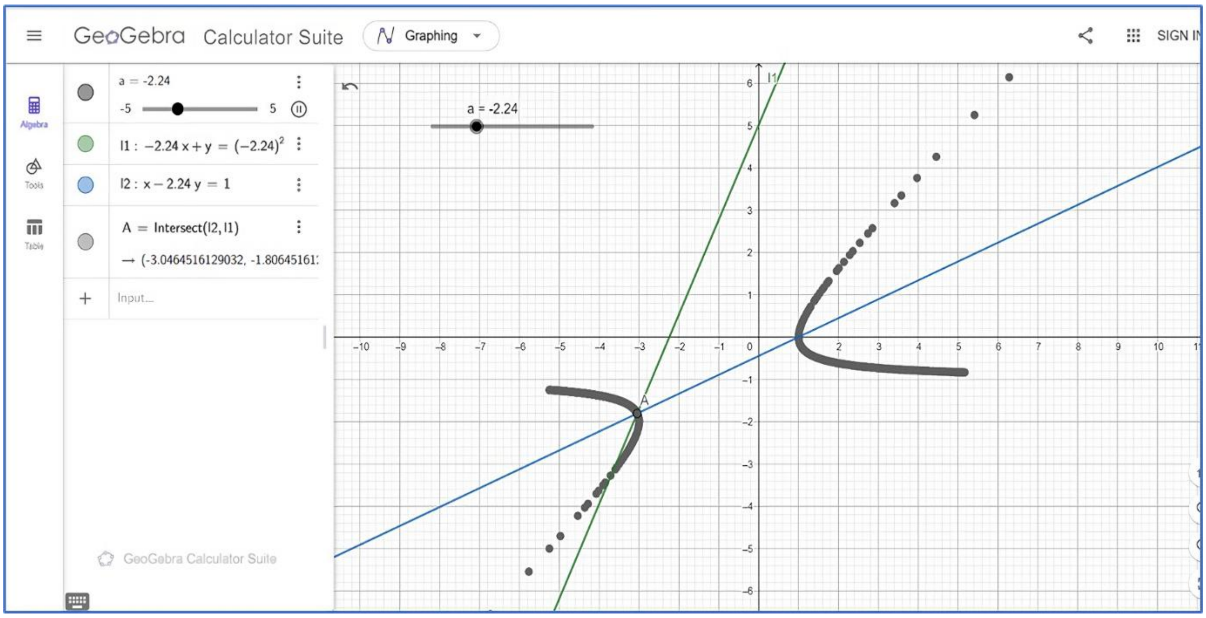

Figure 1. Trace of intersection point of two given lines in DGE GeoGebra (Source: Authors)

For the 2×2 linear system $\begin{cases} ax + y = a^2 \\ x + ay = 1 \end{cases}$ we first calculate the three determinants D, D_x, D_y, as follows:

$$D = \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = a^2 - 1 = (a-1)(a+1)$$

$$D_x = \begin{vmatrix} a^2 & 1 \\ 1 & a \end{vmatrix} = a^3 - 1 = (a-1)(a^2+a+1)$$

$$D_y = \begin{vmatrix} a & a^2 \\ 1 & 1 \end{vmatrix} = a - a^2 = -a(a-1)$$

If $a \neq 1$ and $a \neq -1$, then $D \neq 0$ and the system has a unique solution:

$$x = \frac{D_x}{D} = \frac{(a-1)(a^2+a+1)}{(a-1)(a+1)} = \frac{a^2+a+1}{a+1}$$

$$y = \frac{D_y}{D} = \frac{-a(a-1)}{(a-1)(a+1)} = \frac{-a}{a+1}$$

So, for $a \neq 1$ and $a \neq -1$ the lines l_1, l_2 intersect in the (variable) point

$$A\left(\frac{a^2+a+1}{a+1}, \frac{-a}{a+1}\right)$$

If $a = 1$, then the system turns into $\begin{cases} x + y = 1 \\ x + y = 1 \end{cases}$, which means that the two lines coincide.

If $a = -1$, then the system turns into $\begin{cases} x - y = -1 \\ x - y = 1 \end{cases}$, which means that the two lines are parallel.

The solution expected is completed here. However, one can solve the above textbook exercise by blindly following memorized rules of algebra, without realizing the geometric interpretation of the solution he came up with. But what if we wanted to find the locus of the variable point A?

Such a question would provide a different perspective on the original problem and would highlight the crucial contribution of a DGE to the formulation and test of conjectures. For those purposes we are going to find the locus of the variable point

$$A\left(\frac{a^2+a+1}{a+1}, \frac{-a}{a+1}\right), \forall a \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$$

In DGE GeoGebra we sketch the two lines l_1, l_2, and we point out their intersection point A. Then we draw its trace with the help of the slider tool, for any value of the parameter $a \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

As we can assume by observing **Figure 1**, (which can be found at the following link https://www.geogebra.org/m/kewxx87j) the trace seems to draw a hyperbola. But how can we test this conjecture? And besides, what is the equation of this hyperbola, if it really is such?

For the point

$$A\left(\frac{a^2+a+1}{a+1},\frac{-a}{a+1}\right)$$

we pose

$$x=\frac{a^2+a+1}{a+1} \text{ and } y=\frac{-a}{a+1}$$

So, we observe that

$$x+1=\frac{a^2+a+1}{a+1}+\frac{a+1}{a+1}=\frac{a^2+2a+1}{a+1}+\frac{1}{a+1}=\frac{(a+1)^2}{a+1}+\frac{1}{a+1}=a+1+\frac{1}{a+1}$$

and

$$y+1=\frac{-a}{a+1}+\frac{a+1}{a+1}=\frac{1}{a+1}$$

Therefore, we get Eq. (1), as follows:

$$x+1=\frac{1}{y+1}+y+1$$

The last equation describes the requested locus and can be simplified by appropriate change of the coordinate system. The first step to simplify Eq. (1) is to set $x=x-1$ and $y=y-1$, a geometric transformation called "translation", so Eq. (1) becomes Eq. (2), as follows:

$$x=\frac{1}{y}+y \Leftrightarrow xy-y^2-1=0$$

In order to get rid of the term xy, the *x*- and *y*-axes should be rotated through an angle θ. It is well known from analytic geometry that a "rotation" with center $O(0,0)$ and angle θ is achieved with the use of a linear transformation with matrix $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$. Therefore, we put

$$x=Xcos\theta-Ysin\theta \text{ and } y=Xsin\theta+Ycos\theta$$

so Eq. (2) becomes Eq. (3), as follows:

$$(Xcos\theta-Ysin\theta)(Xsin\theta+Ycos\theta)-(Xsin\theta+Ycos\theta)^2-1=0 \ \Leftrightarrow$$
$$X^2cos\theta sin\theta+XYcos^2\theta-XYsin^2\theta-Y^2sin\theta cos\theta-X^2sin^2\theta-2XYsin\theta cos\theta-Y^2cos^2\theta-1=0$$

The locus we hope to arrive at is a hyperbola, a curve that has both a center of symmetry and an axis of symmetry. Thus, Eq. (3) must be verified simultaneously by pairs of points of the form (X,Y), $(X,-Y)$, $(-X,Y)$, and $(-X,-Y)$, therefore the term XY must be eliminated, i.e., the sum of the factors of XY must be equal to zero. Consequently we have:

$$cos^2\theta-sin^2\theta-2sin\theta cos\theta=0 \Leftrightarrow cos2\theta=sin2\theta \Rightarrow 2\theta=\pi/4 \Leftrightarrow \theta=\pi/8$$

For $\theta=\pi/8$ we get:

$$sin^2\frac{\pi}{8}=\frac{1-cos\frac{\pi}{4}}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4}$$

and (from the fundamental trigonometry identity)

$$cos^2\frac{\pi}{8}=1-sin^2\frac{\pi}{8}=\frac{2+\sqrt{2}}{4}$$

So, Eq. (3) becomes Eq (4), as follows:

$$\frac{\sqrt{2}}{4}X^2-\frac{\sqrt{2}}{4}Y^2-\frac{2-\sqrt{2}}{4}X^2-\frac{2+\sqrt{2}}{4}Y^2=1 \Leftrightarrow$$
$$\frac{\sqrt{2}-1}{2}X^2-\frac{\sqrt{2}+1}{2}Y^2=1 \Leftrightarrow$$
$$\frac{X^2}{2(\sqrt{2}+1)}-\frac{Y^2}{2(\sqrt{2}-1)}=1$$

The last equation is indeed an equation of hyperbola, which has point symmetry about the origin of the axes and parameters

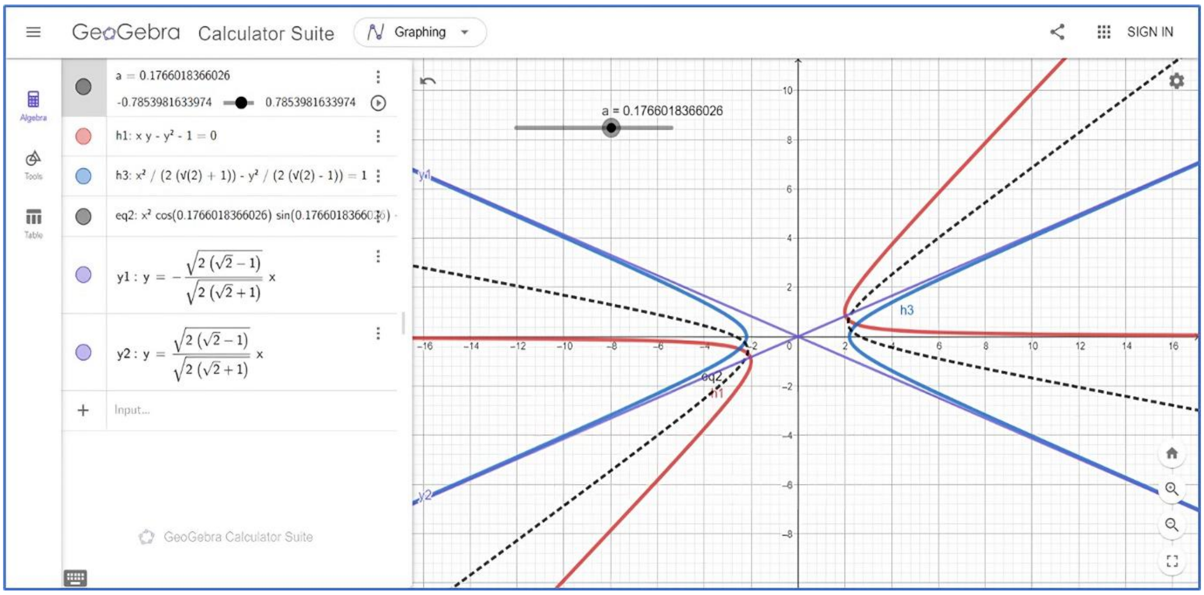

Figure 2. Simulation of rotation of hyperbola (2) by an angle of π/8 in DGE GeoGebra until it coincides with hyperbola (4) (Source: Authors)

$$a = \sqrt{2(\sqrt{2}+1)}\ and\ b = \sqrt{2(\sqrt{2}-1)}$$

Therefore, we have tested our conjecture that the trace of the intersection point A draws a hyperbola with the above equation.

Because the locus is a hyperbola, is well known that there are two asymptotes with the following equations

$$y_1 = -\frac{b}{a}x \text{ and } y_2 = \frac{b}{a}x$$

which are shown in **Figure 2** (https://www.geogebra.org/m/ushwurrk) together with the full simulation of the rotation of the hyperbola (2) until it coincides with the hyperbola (4).

Therefore, after applying the parameters that we have calculated before, the equations of the asymptotes become

$$y_1 = -\frac{\sqrt{2(\sqrt{2}-1)}}{\sqrt{2(\sqrt{2}+1)}}x \text{ and } y_2 = \frac{\sqrt{2(\sqrt{2}-1)}}{\sqrt{2(\sqrt{2}+1)}}x$$

TEACHING PERSPECTIVES AND SUGGESTIONS

Some technologies and mathematics software designed for educational purposes has the potential to contribute to the data collection and analysis, to the experimentation and testing of conjectures by students, and may foster dynamic thinking for students solving mathematical problems (Arzarello et al., 2012). At the same time they provide valuable tools such as exploration and visualization (Hanna, 2000; Leung et al., 2013).

In particular, the educational programming language Logo and DGEs (e.g., GeoGebra) play a significant and potentially unique role in supporting innovative learning trajectories, particularly in geometry, helping students experiment with data and the questions of exercises, to explore and experience mathematics (Arcavi & Hadas, 2000).

Thus, students using dynamic geometry software can engage actively with geometric ideas, while with well-designed activities, appropriate tools, and teachers' support, can make and explore conjectures about geometry and can learn to reason carefully about geometric ideas (NCTM, 2000, p. 41). Such an activity, within a DGE, enabling students to be guided in the formulation and exploration of conjectures, could be based on the problem we analyzed in the previous paragraph and could have the following wording:

In your tablet or in your PC open a GeoGebra window with a set of axes. Construct the lines $l_1: ax + y = a^2$ and $l_2: x + ay = 1$ by writing their equations in the "calculator suite", for any value of the real parameter $a \in (-5, 5)$. Using the "intersect" tool, find the intersection point of the above lines and draw its trace. To do that, you can use the "slider" tool, for any value of the parameter $a \in (-5, 5)$.

What kind of line do you think the intersection point of the two lines draws? Write down your conjectures. If the parameter α takes different values e.g., if $a \in (-10,10)$, does the locus of the intersection point change?

Find algebraically (by using geometric transformations) the coordinates of the intersection point of the two lines and test the conjectures you made in the previous question.

Mathematics teacher could involve and support his/her students in the above activity, in an Algebra or analytic geometry course in the last grades of high school (11th or 12th grade) or in the first years of higher education, depending on the curriculum of each country. At the same time he/she can carry out a research in order to investigate

(a) in which ways (e.g., by measuring or dragging) the students make conjectures when they work in a DGE and

(b) to what extent the students recognize the need for rigorous proof construction.

In addition, mathematics teacher can compare the results of his/her research with those of similar ones, which showed that the participants recognized that the conjectures emerged from visualizing the behavior of objects when moving particular parameters and there was a need to justify or support those conjectures (Santos-Tiago et al., 2018).

Once it is quite difficult to draw the trace of the variable point

$$A\left(\frac{a^2+a+1}{a+1}, \frac{-a}{a+1}\right), \forall a \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$$

by hand, we believe that if the tools provided by GeoGebra (or a similar application) are not used, then it will be quite difficult for the students to speculate that the locus of the point A is a hyperbola. At the same time, however, it is rather impossible for them to validate their conjecture without working in the paper-and-pencil environment. So *a combination* of the two environments (digital and paper-and-pencil) is necessary for the development of mathematical understanding in this case, as well as in other similar ones (Komatsu & Jones, 2020). Otherwise, students miss the opportunity to explore, make conjectures and discover the underlying mathematical structures, confining themselves to a "passive" role that privileges the meaningless performance of algebraic operations.

Students have become very familiar with new ICTs and alternative instructional models due to distance learning, which has dominated education in the last three years (Rizos & Gkrekas, 2022; Rizos et al., 2023; Yates et al., 2021). Also results seem to suggest that prospective teachers believe that digital learning will enable them to have a mathematics pedagogical shift to a less formalized method of teaching that is entertaining and interesting rather than rigorous and traditional (Mulenga & Marban, 2020, p.10). Moreover in the classroom, digital technologies environments can reveal unexpected, hidden conceptual difficulties that teachers could not have seen in paper-and-pencil settings (Abboud & Rogalski, 2021). Thus, we consider that it is a chance to integrate in a more creative way technology and especially DGEs in mathematics education in a way that will complement the traditional paper-and-pencil environment, as we saw in the activity above and take the extra step.

The formalistic character of the teaching and learning of mathematics (that is blind memorization of algebraic rules, arithmetic operations and solving techniques without conceptual content and without any geometric interpretation), at least in Greece with which students have been familiar with since the early secondary education (Rizos & Adam, 2022), is inherent in the loss of meaning of mathematical concepts and the ease with which generalizations are attempted, which are not always accompanied by a deeper understanding of their meaning. The result of this situation is the inability of students to understand problems combining algebra and geometry, and their difficulty in relating the elements of these problems to appropriate concepts and procedures, which they can work out. So we suggest that activities that combine algebra with geometry in a technology-enriched learning environment should be discussed more in the classroom. This approach enhances students' ability to explore, reconstruct (or reinvent) and explain mathematical concepts (Dockendorff & Solar, 2018), and has positive effect on students' attitudes towards mathematics, thus enhancing their learning and achievement (Vasquez, 2015).

Of course we do not support the replacement of face-to-face teaching in the class with "black-box Technology" i.e., programs that simulate mathematical constructions and thus do not require their users to understand their underlying mathematical structures (Rizos et al., 2021). In this direction the attention has to be paid to ensure that

(a) users of technology applications are aware of the mathematical structures or mathematical models involved and the key conditions under which they are applied and

(b) Technology should not substitute students' experience in two- and three-dimensional space.

CONCLUDING REMARKS

In this article we sought to present the benefits of the mathematical elaboration of a high school-level (or higher) exercise combining two different approaches and environments: the paper-and-pencil environment and DGE GeoGebra. In addition, we proposed a relative activity that includes several mathematics disciplines, such as algebra, analytic geometry and the integration of ICTs, in a mathematics education context, which may be valuable in a future study. An experiment in educational research could be designed based on the above activity, focused on exploring the ways in which students make conjectures when working in DGEs, and whether they recognize the need for rigorous proofs. After all there seems to be room for conjectures in mathematics. With the appropriate integration of DGEs, conjectures emerge, even though standard school activities. These conjectures could contribute to enhance students' mathematical understanding, actively involve them in the learning process and help them to move from exploration to rigorous reasoning.

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