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Five Years of Comparison Between Euclidian Plane Geometry and Spherical Geometry in Primary Schools: An Experimental Study

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Abstract:
We present the result of an eight-year didactic experiment in two primary school classes involving comparative geometry activities: a comparison between Euclidean plane geometry and spherical geometry that took place over five years. Following the didactic experiment, three years on from the end of the experiment, final questionnaires were administered and codified in order to evaluate the project's effect on the pupils' school performance and attitude, especially with regard to mathematics.
Keywords: comparative geometry, Lénárt spheres

INTRODUCTION

This research is part of an international framework focused on the value and effectiveness of the introduction of spherical geometry, or rather comparative geometry within Euclidean geometry teaching practices (Lénárt, 1993). The awareness that mathematics is regarded as a difficult school subject requires, among other things, a transformation of the teacher's way of teaching (possibly with the use of other tools) to increase students' interest and active participation in the classroom (Gambini, 2021). One of the major problems in learning in the school context is, in fact, to motivate students to engage fully (Stipek et al., 1998).

The idea for this study arose from the idea that a comparative approach to geometry, i.e., the comparison between Euclidean plane geometry and another geometry, can on one hand help in understanding the fundamental elements and properties of figures in plane geometry, making it possible to distinguish their properties from superstructures (Sbaragli, 2005) and on the other hand motivate pupils by challenging them on their perception of the nature of mathematics, particularly geometry.

Comparative geometry allows students to develop geometrical concepts from concrete experiences and objects; to develop specific skills related to thought processes typical of geometry and mathematics; to operate and communicate meanings with specific languages, and to use these languages to represent and build models; and to communicate and discuss, to argue, to understand the points of view and arguments of others (Lenart, 2007).

The idea is to teach two or three types of geometry at the same time, continuously comparing and contrasting the different forms. We started with the plane and spherical surface because these surfaces are familiar from primary school and everyday experience. Later, one could add hyperbolic geometry,

as different concepts about the plane and sphere can only really be understood when compared with this third type of geometry (Kotarinou & Stathopoulou, 2017).

In recent years, alternative approaches in the teaching of geometry have been studied. The use of new technologies (Jones, 2011; Laborde et al., 2006; Oldknow, 2008) has allowed students to become involved in the teaching/learning process by creating situations.

We also evaluate the integration of all available resources and techniques as an enrichment of mathematics teaching and at the same time we challenge students' perception of the nature of mathematics. In this context, students face the challenge of seeing mathematics as a continuous spectrum that penetrates various aspects of their lives both now and in the future, impacting on both individual and social needs. Speaking specifically about geometry, traditional teaching methods have never proved particularly successful.

The main focus of the contribution in question is a qualitative survey carried out with students at the end of middle school (grade 08) who participated in the experimental project in comparative geometry during primary school: we want to clarify that the questionnaire and the interviews were administered after three years from the end of primary school and therefore after three years from the end of the experiment. During the experimental project, non-standard situations (Baldazzi et al., 2013) were proposed to children of two sections, from grade 01 to grade 05, in which the exploration of problematic situations in spherical geometry was often also a means to acquire the skills and knowledge of the area of Space and Figures as established in the 2012 Italian National Guidelines (MIP, 2012).

Following the questionnaire, the pupils' answers were divided into three categories that correspond to their perception of how much the experiment has: *increased their competence* in mathematics, *improved their vision* of mathematics (in terms of mathematical education and lessons of mathematics) and i*ncreased their motivation and interest* in mathematics.

THEORETICAL RATIONALE

One of the goals was to design a vertical path for primary school based on observation, analysis of analogies and comparison between some of the main concepts of plane geometry and spherical geometry (distance, angles, area, lines, basic plane shapes). The first two years of the experiment were the subject of in-depth studies (Bolondi et al., 2014) which continued over the following three years.

This involves a radical innovation of the discipline; the "monothematic message" perceived by standard classroom practices is transformed into a dynamic apparatus based on the "dialogue" between two or more different systems (Antonini & Marracci, 2013). On the other hand, "Geometry starts from the spatial, visual, and tactile experience (seeing and touching objects), or even motor (we move between objects and move them)", see Speranza (1988).

The didactic proposal is innovative, complex and structured, and has allowed us to further investigate some aspects related to study of the plane through a laboratory activity on the spherical surface, also with interdisciplinary feedback on geography, history and art. Numerous research studies show that a didactic proposal of this kind is functional in the process of teaching/learning geometry because it offers students different approaches to the same theme (e.g., Lénárt, 2007).

The tools used directly involve use of the body, structuring the individual's action and orienting the perception. Such a tool "incorporates" certain collective knowledge and experiences which "guarantee" its functioning (Antonini & Marracci, 2013).

Sensory-motor experiences are fundamental for the formulation of even abstract concepts of mathematics: doing, touching, moving, and seeing are essential components of mathematical thought processes (Gallese & Lakoff, 2005).

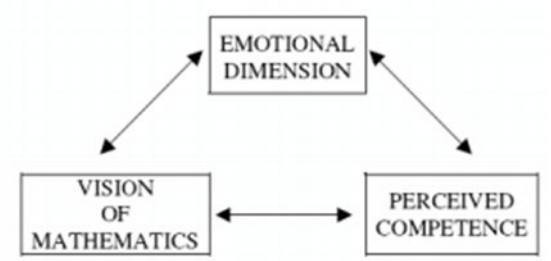

Figure 1. Strictly interconnected dimensions

The added value of this work is the questionnaire administered to students at the end of middle school (2019), who had started the course in 2011 at primary school, to assess the impact that the teaching of comparative geometry had on their school view, particularly in the approach to middle school geometry where no reference to spherical geometry was made. The questionnaire was accompanied by some interviews with the children to have a more complete picture of their experience.

Up to now, improvement from a didactical point of view in activities of this type has been evaluated only in the short term, immediately after the activities were carried out, while in our case the evaluation was made after three years with students who (at the time of the questionnaire) were no longer attending their school of origin and who had grown both from a cultural point of view and from the point of view of cognitive development.

The final questionnaire is evaluated based on attitude toward mathematics. We follow the TMA model introduced by Di Martino, Zan (2010) characterized by three strictly interconnected dimensions (See **Figure 1**):

- emotional disposition toward mathematics
- vision of mathematics
- perceived competence in mathematics.

Our research hypothesis relies on the fact that transition from one form of pairing to another is an important step in the learning process. This can be a result of a conceptual change (diSessa, 2006). We adopt Duval's point of view that there are 4 levels of understanding of a geometric figure (Duval, 1995, 1999): the passage from one level to another is in fact the result of a conceptual change: representation and visualisation are at the core of understanding in mathematics, in fact, representation refers to a wide range of activities of meaning, various ways of evoking and denoting objects, the way information is coded (Duval, 1999). In fact, geometry captures and formalises some aspects of our daily sensory-motor experience which are related to "spatiality".

Visualisation and representation are processes that play a fundamental role in the learning process of mathematics and, even more so, in cognitive architecture related to the comprehension of geometric concepts.

Therefore, in geometry it is necessary to combine the use of at least two systems of representation, one for verbal expression of properties or numerical expression of magnitude and the other for visualisation. A "geometric shape", as it is called, always associates both discursive and visual representations, even if only one of these can be explicitly highlighted according to the mathematical activity required.

The progressive fusion of conceptual (in terms of identification and use of geometric properties) and figural aspects (in terms of properties in representations) is made explicit by children through language acquisition shapes, in a Vygotskian perspective (Vygostki, 1978).

A drawing acts as a geometric shape when it activates the level of perceptual understanding and at least one of the other levels. The perceptive level involves the ability to recognise figures (e.g., distinguish

shapes) and to identify the components of a figure (recognise sides or other elements). The epistemological function of the perceptual level is identification.

We therefore asked ourselves the following research questions.

What experiences can be achieved via a sphere, and what reflections can be promoted in this regard for primary school pupils?

Is it possible to build a solid foundation in geometry that remains over time, using comparative geometry in primary school?

Does a non-Euclidean path of geometry in primary school have a positive impact on the study of geometry in subsequent school levels?

Comparative Geometry

As previously mentioned, our methodology for intervening in these interactions is based on the use of comparative geometry. The idea of comparative geometry is to compare the basic concepts of spherical geometry with the corresponding ideas of plane geometry, highlighting similarities and differences. The sphere is not a foreign object even for a primary school student and this approach offers students and teachers the opportunity to learn how to achieve creative thinking by discovering a new geometry. The added value of a primary school student is the fact that they are not yet influenced by several years of studying Euclidean geometry, making their propensity to explore non-standard situations more effective (Lénárt, 1993).

Children's learning is in fact always situated learning: that is, if we build a learning environment of a certain concept, children will learn that concept but codify it solely to that environment (which we generally call "artificial learning environment"). The naive dream that children could learn in an artificial environment and could consider using this learning in any situation, in a kind of spontaneous cognitive transference, is and remains a utopia.

Comparative geometry activities therefore allow children to deal with geometrical objects in a learning environment where the relationships between objects, representations and properties are different from the usual ones, which implies a restructuring of the interactions.

In this experimentation, some topics were introduced first in spherical geometry and then in plane geometry: for example, the concept of circumference was introduced in the second class of primary school, deviating from the norm (fourth or fifth year of primary school in Italy) and was introduced as a circumference on the sphere. The results of this first phase (Bolondi et al., 2014) have shown how students associate a content to the word "circle" in different ways, which refer to different processes in which the interactions between objects, their properties and their representations can change due to a didactic action. Categories have been used to classify the behaviour of the entire research population, with the aim of mapping the evolution of these processes throughout the different school levels.

The Lénárt Sphere

The exploration of spherical geometry requires drawing shapes on a spherical surface because it is not enough to imagine a spherical shape drawn on a plane. The Lénárt Sphere kit helped us to create a new learning environment to make geometry using a plastic sphere, markers, a spherical ruler and a spherical compass.

Lénárt Spheres (Lénárt, 1993, 1996) are a well-known tool used to provide a learning environment for comparative geometry activities where the relationships between points, circumferences, right angles, properties such as minimum distance and so on are different from the usual ones. They are used at all school levels. Although used mainly in advanced mathematical teaching (especially for the exploration of non-Euclidean geometries), similar artifacts have already proved useful to investigate children's

actions and their construction of mathematical meanings (Antonini & Marracci, 2013) because the comparison of geometries can help teachers to create a-didactic situations, students to rid themselves of some mathematical preconceptions and prejudices, and both teachers and students to break the didactic contract (Brousseau, 1997).

Our hypothesis is that experimenting with spherical geometry activities in tandem with those of plane geometry helps students to better understand geometrical objects and increases students' intuitions about plane geometry to such an extent that a positive effect is generated on subsequent school grades about the study of geometry.

The surface of the sphere, as an example of a limited surface, an alternative to the plane characterised by limitlessness, allows children to become aware of a two-dimensional space, with characteristics different from the plane, in which they can experiment topologically paths, construct figures and measurements, use and expand spatial visualisation, and try their hand at the construction of artifacts and the use of more technical tools.

Geometry in these contexts is frequently encountered with geography: since our planet has a shape that can be compared to a sphere, we move on it to travel and go from one place to another. Here Euclidean and plane geometry is no longer applicable; it is unusual to speak of the shortest path between two points on the globe and the concept of "distance" is only addressed in the plane. However, in the experimented teaching path, problems emerge that make children and teachers themselves reflect on the differences and similarities that the same concept assumes in the two worlds: that of the plane and that of the sphere.

For example, comparison between plane and spherical geometry offers a great opportunity to raise questions about the concept of circumference and the relationship between circumference and straight line: the "simplest" line in spherical geometry is a maximum circumference (the great circle), the largest circumference that can be drawn on a sphere, for example the line of the Equator or the meridians (on the surface of the Earth). However, a great circle can also be considered a "straight line" on the sphere, in addition to being a circumference. Other circumferences, e.g., parallels, excluding the Equator, do not play the same role as "straight lines" on the sphere.

Geometric concepts on the sphere, and related activities, are no more difficult than those of plane geometry, even for a primary school student: the word "circle" is familiar to our students, both through school activities and through natural language.

METHODOLOGY

The Research Design

In this paper, the qualitative research design is applied in order to analyse the effect of comparative geometry. Within qualitative research methodology, the goal is to obtain "in-depth information" by collecting very detailed data using both questionnaire and interviews.

In this context, semi-structured interviews are useful for data collection, in fact, according to Yildirim and Simsek (2005), it is possible to determine experiences, attitudes, opinions, and mental perceptions through interviews. Therefore, semi-structured forms of interviews were developed in addition to questionnaires. Each approximately 10-minute interview with participants was recorded and conducted in a single session, however the students are minors than names of participants are not revealed.

In the first and second years the children took an interdisciplinary course involving mathematics, fairy tales and history. The experimentation began with the analysis, observation, and verbalisation of the characteristics of the solid figures (cubes, spheres, pyramids, ...) and then observed their nets on the plane. Then, they were tasked with the construction of a model composed partly of a plane and partly curved surface, of which maps, and paths were drawn, and paths executed, focusing on the concept of

Figure 2. Measuring distances on a planisphere

Figure 3. Spherical triangles

distance between two points. Secondly, the children observed "the terrestrial globe", working to find the minimum path connecting two points on both maps and globes. The problem allowed them to compare the lengths of the paths, first evaluating them by eye, then devising simple tools and measuring systems. The concept of straight line and segment as "shortest path between two points" gradually matured, with different moments of comparison between conflicting opinions, verification, mutual conviction and discovery (See **Figure 2**).

Experiences are made to acquire the concept of great circle, i.e., the "straight line" on the spherical surface: rolling balls along a straight trajectory, dropping a drop of water on the spherical surface, or passing rubber bands around spheres without them slipping away.

On the plane we can find straight lines that do not meet, whereas on the sphere the straight lines always meet in two points, which are "opposite" each other.

In the third-year classes the children recognised the angles and understood that in the plane they are characterised by 'unlimitedness'; on the other hand, on the sphere they are well defined regions. Using the ruler and compass of the Lénárt kit, they measured the angular quantities. The pupils discovered that on the sphere, unlike in the plane, there are no parallel lines; there are perpendicular lines between them as in the plane (but these can be more than two by two perpendiculars), and the segment on the sphere is part of a great circle. The experience of spherical geometry allowed children to better understand the concepts of latitude and longitude, helping them to avoid forming the misconception that the amplitude of an angle is the one commonly marked with a bow as opposed to the whole part of the plane of the spherical shape (See **Figure 3**).

In the fourth-year classes the interdisciplinarity with history and geography led to an involvement of the voyages of the great navigators. For example, in one class Magellan's voyage was proposed, taking inspiration from the book "Magellano e l'Oceano che non c'era" of L. Novelli ("Magellan and the Ocean that was not there"), about the daring and fatal voyage of explorer Ferdinand Magellan, who first managed to circumnavigate the Earth thereby showing that its shape was spherical - the teacher

Table 1. Resume of the activities

Grade	Activities
1	Introduction with fairy tales, riddles and history; verbalisation od geometric shapes and solids; straight lines, shortest paths, distance, measures.
2	Angles; geometric shapes on the spherical surface: triangles, biangles, circles. Properties of figures.
3	Measure of angles, parallels and perpendicular; comparison with plane; limitedness of the sphere. Activities on the globe, longitude and latitude.
4	History and geography: voyages of the great navigators; quadrilateral and other shapes on the spherical surface.
5	Tessellations of the spheres, Platonic solid.

together with the children reconstructed the various stages. Using a wire, the children visualised the itinerary of the circumnavigation of the Strait that today bears his name.

The route also continued inside the geometry discovering the polygons on the spherical surface: the children discovered that on the sphere it is also possible to have spherical bilateral polygons, while on the plane it takes at least three sides to close the geometric shape. The children built the triangles and discovered that on the sphere the sum of their internal angles can have an amplitude greater than 180°. On the plane we can construct the square, whereas on the sphere this is no longer possible - we can only construct a quadrilateral with equal sides, but this does not have angles of 90°.

In the fifth-year classes, several topics were dealt with, including Platonic tessellations and solids: tessellations were dealt with both on the plane and the sphere. Children discovered that it is possible to tessellate the plane with some regular polygons: the equilateral triangle, the square and the hexagon; the necessary condition is that the measurement of the angles is 360°. The construction of the Platonic solids to be inscribed in the Lénárt Sphere was a decidedly manipulative activity. The children built the cube, the octahedron, the tetrahedron, the dodecahedron and the icosahedron, before inserting each polyhedron into an acetate sphere. With a marker pen they marked the position of the vertices that touched the surface of the sphere and with the spherical ruler they traced the segments that joined the vertices. The children discovered, among other things, that hexagons cannot tessellate a spherical surface.

During the various phases of experimentation, the experiences were structured in such a way as to alternate moments of practical activity, observation and manipulation, with moments of reflection and re-elaboration of shared activities, so that the children became aware of the similarities and differences existing between the plane and the spherical surface. By immersing themselves in a world with characteristics different from the plane, they were able to experience paths on spherical surfaces, build figures and make measurements. By using and expanding spatial visualisation, the students were able to construct artifacts and use more technical tools than those used in standard teaching activities. Throughout the experiences, the various problems and discoveries that arose through discussion and comparison between the new knowledge and the various skills acquired were addressed in the community; an attempt was made to "monitor" the various images and mental models that were gradually developing in the geometric field (D'Amore, 1999).

Our planned activities with the Lénárt Spheres are designed for small cooperative groups. Working in small groups gives the best results: each group worked on a sphere placed in the centre and three or four students around it. Each student could have easy access to the sphere and all the other tools on the table. The tables and chairs were arranged so that the student could see the tools on the table and his/her peers around the table. However, there was no lack of activities throughout the class - half the class participated in activities as well as discussions with peers and teachers.

The Population

The experiments took place at the Istituto Comprensivo "De Andreis" in Milan and the Istituto Comprensivo "Montanari" in Ravenna and were conducted by Rosalia Tusa and Lucia Baldazzi with the collaboration of a group of researchers.

The pupils were starting primary school, 31 in one class and 26 in the other (a total of 25 girls and 32 boys). However, not all children participated in the final questionnaire after the third year of middle school. There were 11 participants in the final questionnaire for the school in Ravenna and 30 for the school in Milan. The data were reported aggregated.

The percentage of non-native speakers was about 15%, classes were identified after discussions with teachers and head teachers. These classes were of medium-high level, compared to the Italian average level, both in terms of socio-economic background and learning performance (the comparison was made through the results of national INVALSI standardized tests).

The teachers participated in the activities of a research group in mathematics teaching. The teachers were not initially prepared on non-Euclidean geometry and designed the activities together with researchers from the University of Bologna. The meetings took place about once every two weeks and at least once a year a researcher entered the classroom to accompany the teacher. Lessons were always held during school hours except for rare occasions.

Task-based Interviews, Semi-structured Interview

At the end of both courses, a questionnaire was given to the students, (now attending Middle School) aimed at investigating their beliefs on the experience both in terms of attitudes towards mathematics and in terms of personal growth from the point of view of content.

The proposed questionnaire consists of open-ended and closed questions on Likert's scale. 41 pupils from the two previous primary school classes participated in the survey. In some cases, an affirmative answer required further argument or explanation.

The proposed questionnaire was as follows:

1. In your opinion was the work with Lénárt Spheres useful in your studies in middle school? Why?

   1a. If so, can you specify in which situations it has been useful to you?

2. What did you find out about the activities you did?

3. Would you like to redo some activities with Lénárt Spheres on other concepts of geometry?

4. Do you think that the activities with Lénárt Spheres have helped you to understand some concepts of geometry?

   4a. If so, which ones?

5. Would you like to do maths activities on other mathematical content like those done with the Lénárt Spheres?

6. **Table 2** shows some statements. Please state your level of agreement from 1 to 4 for each of the following statements.

Table 2. Question number 6

	1 – Not at all	2 – A Little	3- Quite a lot	4 – A lot
a. During maths class I have fun				
b. During maths classes I feel like I am learning the concepts/contents				
c. During maths lessons I am attentive, and committed only because "I have to be so"				
d. During the activities with Lénárt Spheres I had fun				
e. During the activities with Lénárt Spheres I feel like I am learning the concepts/contents				
f. During the activities with Lénárt Spheres I was attentive and committed only because "I had to be so".				

Table 3. Question number 7

	1 – Not at all	2 – A Little	3- Quite a lot	4 – A lot
a. During maths class I have fun	7	11	18	5
b. During maths classes, I feel like I am learning the concepts/contents	4	14	12	11
c. During maths hours I am attentive, and committed only because "I have to be so"	14	7	18	2
d. During the activities with Lénárt Spheres I had fun	2	0	7	32
e. During the activities with Lénárt Spheres I felt like I was learning the concepts/contents	0	1	16	24
f. During the activities with Lénárt Spheres, I was attentive and committed only because "I had to be so".	21	13	3	4

Table 4. Question number 8

Did you happen to tell anyone about the activities done with Lénárt Spheres? To whom and what did you say?	Yes 26 To whom? • Parents and my sister. • to my parents • to my mother • to my family • to relatives • to my friends • to the teacher What did you tell them? • Everything we were doing in the lab with the spheres... About the giant globe because we had to do a task at school...	No 15 (One of the pupils' answers: "No, because we haven't done everything yet, but I'll tell the teacher later").

7. Some statements are shown in **Table 3**. Please state your degree of agreement from 1 to 4 for each of the statements in **Table 3**.

8. Did you happen to tell anyone about the activities done with Lénárt Spheres? To whom, and what did you say? (See **Table 4**).

DATA PRESENTATION

The results of the interviews are presented in **Table 5** in an aggregated manner and are analysed in Section *Discussion*.

Table 5. Aggregated questionnaire data

NUMBER OF PUPILS PARTICIPATING IN THE SURVEY: 41			
Questions	Yes		No
1. In your opinion was the work with Lénárt Spheres useful in your studies, in middle school? Why?	26		14
1a. If so, can you say in which situations it has been useful to you?	When I compared: • Straight lines • latitude and longitude • flat figures • circumference and semicircle • Orthogonal projections • In other subjects of study • In geography (meridians, parallels, equator) • In geometry (circumference, diameter, radius)		(Because we have not yet studied the sphere but in geography, yes, when we studied the meridians and parallels)
2. What did you find out about the activities you did?	Activities concerning the acquisition of the concepts of: • latitude and longitude • maximum circumference • opposite points • distance • Build Platonic solids and inscribe them in the spheres • Tassel the spherical surface • Use the protractor, ruler and spherical compass • Snow White and the Seven Dwarfs • Exhibit at the Rotonda della Besana (Milan) where we described to other students and their teachers our experiences of geometry activities in the sphere. • Building polyhedra like the cube and the pyramid to inscribe in the spheres • Activities with oranges • Use instruments such as the protractor, ruler and spherical compass • Draw on the sphere, draw lines on the sphere, draw circumferences and radii • The ant's paths on the sphere • The bear problem • Snow White and the seven dwarves • The giant globe • The puzzle globes		
3. Would you like to redo some activities with Lénárt Spheres on other concepts of geometry?	Yes 35	No 2	Probably 4
4. Do you think that the activities with Lénárt Spheres have helped you to understand some concepts of geometry?	30	5	6
4a. If so, which ones?	• Straight lines, longitude and latitude, flat figures, circumference and semicircle, orthogonal projections. • certain terms I understood thanks to the work with the sphere • how to measure with instruments for flat or curved figures • the radius, diameter, circumference • quarters of sphere • to represent the geometric shapes		
5. Would you like to do maths activities on other mathematical content such as those done with Lénárt Spheres?	Yes 36	No 5	Probably

DISCUSSION

The analyses of the qualitative survey show positive results both in terms of beliefs about the experience regarding attitudes towards mathematics and in terms of personal growth from a content point of view (see Di Martino & Zan, 2010).

As we mentioned in the introduction, we divide the students' answers into three categories:

- perceived competence in mathematics,
- vision of mathematics,
- emotional attitude and motivation toward mathematics,

and we see how the experimentation carried out in primary school may have improved one or more of these categories.

From the point of view of increasing knowledge and competence in mathematics, very positive feedback emerges from the answers given by the students (questions 1, 2, 4): 26 students out of 41 interviewed state that the experimentation was also useful in middle school and 31 students state that the activities with the Lénárt Sphere helped them to understand some more concepts of plane geometry (see Lénárt, 1996, 2007). These statements and other evidence emerging from the analysis of the questionnaires show positive impacts and spin-off effects regarding the learning process of some geometric concepts.

Some of their answers show that their competences in the field of geography, in the study of terrestrial coordinates, benefited them enormously. On the other hand, one can speak of cultural growth also within geometry itself, especially in the study of the circumference and its properties.

Students recalled here many of the activities that were done during the experiment. Longitude and latitude and paths along arcs of maximum circumference were introduced during the phase where Earth geography was introduced. Cartography was also discussed at this juncture; in fact, the students also remembered something about map projections. The part about tessellations and polyhedral was done during the fifth-class experimentation.

The snow-white activity is described below while the bear problem is still a spherical geometry problem: "A bear leaves its den, travels 10 km south, then 10 km west and finally 10 km north and returns to the starting point. What colour is the bear?" Starting with this riddle, the kids experienced the "oddities" of the new geometry.

The kids refer to the path of the ant that is immersed in a two-dimensional surface but does not know if it is on a flat or spherical surface. A few activities were done to allow the hypothetical ant to figure out what surface it was on.

Question 6 concerns their vision of mathematics: compared to traditional mathematics lessons, 39 students say they had fun in comparative geometry activities, 40 had the feeling they were learning new concepts and only 7 say they were paying attention just because they had to.

In contrast, for the other mathematics activities in the classroom the number of students who say they had fun is 23, as well as those who claim to have learned new concepts, while 20 claimed to be attentive only because they had to be. Considering that the classes are of medium-high level (gauging by the results of the national INVALSI standardized assessments), the activity had a very positive effect.

Finally questions 3, 5, and 7 concern their emotional and motivational point of view: almost all the students interviewed (36 out of 41) stated that they would like to do mathematical activities on other mathematical contents such as those done with the Lénárt Sphere and 35 stated that they would like to do further activities with the Lénárt Sphere on other concepts of geometry. Many of them remember very well particular activities carried out in the spherical geometry workshops (activities carried out

from 3 to 8 years before): inscribed polyhedra, activities with oranges, the globe puzzle, the game of Snow White and the Seven Dwarfs (the game consists of dividing the sphere into 8 equal sectors and placing Snow White and the Seven Dwarfs: the other children, without seeing the sphere and only able to ask questions like "Is Snow White near Happy?" or "Is Sleepy opposite Sneezy?", have to guess the exact configuration), tessellation etc... and most of the pupils shared these experiences with family and friends. Some of them also remember sharing their experiences at local science festivals.

Moreover, 26 out of 41 state that they have talked about the activity with parents, relatives, and friends and some also with the secondary school teacher. Of the remaining 15 who did not talk about it, some state that they will tell their teacher as soon as they address some of the topics (e.g. geography) dealt with during the experiment.

The experiments were structured in such a way as to alternate moments of operation, observation and manipulation, with moments of reflection and collective re-elaboration of the activities, in order to highlight the similarities and differences between flat and spherical surfaces. Immersing themselves in a world with characteristics different from the plane, the children were able to experiment with paths, build figures and make measurements. The discussion about the comparison between the different systems allowed to stimulate conjectures, arguments and simple demonstrations among the pupils. Questions and problems were addressed in groups, discussing, and communicating also with teachers, to transform the knowledge that was gradually building into shared knowledge.

The proposal of multiple activities of manipulation of some spherical objects made the spherical surface an object with which even the young children could work actively, creating a rich and meaningful geometric learning environment. A similar goal was achieved in the research of Antonini and Marracci (2013).

We believe that the most important result achieved has been the multimodality of teaching, the contemporary use in class of mathematics and laboratory tools. The use of a particular artifact has allowed children to investigate again the relationships between mathematical objects, their representations, their constructions, and their properties. After the use of the artifact for instance, 11 children out of 46 changed their way of describing a circle (Bolondi et al., 2014).

CONCLUSION

We argue that comparative geometry can be taught to the advantage of all school grades and in general can also help the student in other areas of mathematics.

There are several experiences supporting this from primary school to university and in teacher training, with a considerable amount of experimental evidence, both positive and negative.

Our aim has been to give a clearer understanding of geometric concepts, directed by research and discovery in mathematics, so that children gain self-confidence - the satisfaction experienced in these activities has had a positive effect even after years of experimentation.

This approach, open to the emotional, physical and intellectual commitment of the students, has inspired their greater participation in mathematical thought and expression and has led them to a deeper understanding and appreciation of mathematics.

One of the critical issues and motivation for further investigation is the way the questionnaire and interviews were set up: it made little sense to administer it to the control group as well. However, the idea is to continue with the research in this direction, taking also into account school performance of each student.

The experimentation in fact confirms the results obtained by Lénárt (1993) also for the elementary school in which many topics were touched upon and many discussions arose on the subject. To give an

example, the concept and the definition of circumference have been outlined since the second class through a discussion with the children that allowed them to define the concept independently from the surface on which they were working.

The most interesting part, however, relates to the other two research questions because an analysis of the questionnaire and the interviews verified that comparative geometry experiences remained over time and allowed for a better consolidation of geometric concepts in later school grades, as reported in the responses to question 4 of the questionnaire.

We did not conduct specific interviews with teachers but interviews with them revealed points of view the experience was positive and informative in line with what we found for preservice teachers in (see Gambini & Lénárt, 2021): the primary school teachers who took part in the experiment had no background in spherical geometry and this experience firstly allowed them to compare notes together and secondly to encourage the children in a comparative approach, who were so enchanted by the new geometry that they were not willing to accept the geometry of the plane again.

The teachers recognized the advantages of this method, not only to arouse and strengthen interest in geometry, but also to improve and speed up understanding of Euclidean concepts.

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