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Research Article

Exploring the mathematics teacher's specialized knowledge through the sigmoid function

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ABSTRACT

This study explores the specialized knowledge mobilized by a mathematics teacher when implementing an interdisciplinary activity based on the sigmoid function in an 11th grade class. The research aims to identify the mathematical and didactic knowledge elements present during the activity and to detect potential conceptual or pedagogical limitations in the teacher's practice. A qualitative instrumental case study was conducted within an interpretive paradigm, using non-participant classroom observations and reflective field notes as primary data sources. The intervention was conducted in a public high school in Madrid and involved fifteen science-track students across two 50-minute sessions. It consisted of a complete mathematical study of the sigmoid function and reflective reasoning tasks designed to contextualize the concept within real-world applications. The analysis was framed using the mathematics teacher's specialized knowledge model. The findings reveal that the teacher demonstrated strong competencies in several subdomains of this model, particularly in content knowledge, inter-conceptual connections, and pedagogical strategies. Additionally, the teacher showed strong awareness of student thinking and provided targeted instructional support. However, the study identified minor limitations in the teacher's epistemological contextualization of the mathematical content beyond its technical aspects, pointing to the need for deeper engagement with its historical and conceptual foundations. Overall, the study confirms the usefulness of the model in characterizing the complexity and integration of knowledge required for teaching interdisciplinary mathematical content effectively, although it would be necessary to refine certain knowledge elements in future work.

Keywords: mathematics teacher's knowledge, specialized knowledge, teaching and learning of mathematics, sigmoid function

INTRODUCTION

The study of mathematics teachers' knowledge has undergone a continuous process of reflection since Shulman (1986) became interested in understanding how teachers transform their professional knowledge into purely didactic knowledge, with the purpose of making it more understandable to students (Caviedes Barrera et al., 2023). In his work, Shulman (1986) emphasized the importance of approaching this knowledge from a disciplinary perspective, organizing it into three basic domains: disciplinary knowledge, pedagogical content knowledge (PCK), and curricular knowledge.

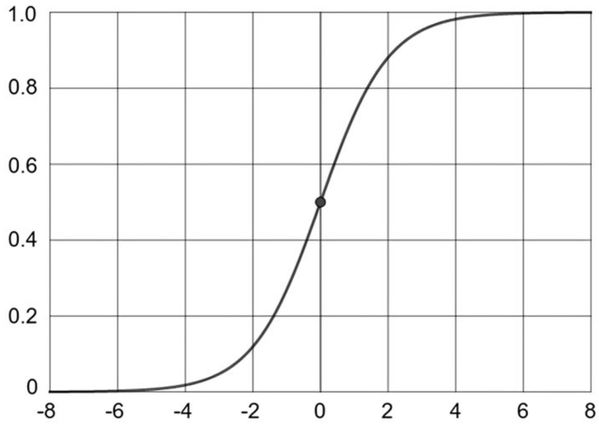

Figure 1. Sigmoid curve: $y = \frac{1}{1+e^{-x}}$ (created and formatted with GeoGebra)

Over the years, different theoretical and methodological approaches have been proposed to further explore the elements that should be part of this knowledge in mathematics didactics (e.g., Ball et al., 2008; Baumert & Kunter, 2013; Carrillo et al., 2013; Godino, 2009; Ponte, 1994; Schoenfeld & Kilpatrick; Tatto et al., 2008). These approaches analyze the structure that mathematics teachers' knowledge should have, highlighting the factors involved in the teaching-learning processes of the discipline and providing teachers with resources to organize their teaching activities (Meléndez-Cruz et al., 2023).

One of the most relevant contributions was made by Ball et al. (2008) in their theoretical model *mathematics knowledge for teaching* (MKT), in which they included *specialized content knowledge* to differentiate it from the common knowledge that any person with a basic mathematical background might have. For their part, Carrillo et al. (2013), assuming this specialization, sought to identify which elements of the mathematics teacher's knowledge are necessary to promote reasoning, argumentation, refutation, representation, or modeling of knowledge toward constructive learning (Muñoz-Catalán et al., 2015). This implies not only delving into purely formal mathematical concepts but also into their origins, epistemological foundations, and relationships with other concepts. Thus, in order to redefine some aspects of the MKT model, they created a new knowledge model called *mathematics teacher's specialized knowledge* (MTSK). This model encompasses the set of mathematical and didactic knowledge specific to the mathematics teacher (Carrillo-Yáñez et al., 2018). However, mathematics encompasses various areas of knowledge that are typically used in an interdisciplinary manner with other scientific, technological, humanistic, and even artistic fields, to address a specific phenomenon or problem. In fact, one of the challenges mathematics teachers face today is providing students with interdisciplinary training that allows them to contextualize mathematics and connect it with the real world (Zamorano et al., 2019). This encourages, among other things, meaningful learning, critical and creative thinking, logical reasoning, and students' ability to abstract (Perignat & Katz-Buonincontro, 2019).

In this regard, an interdisciplinary activity has been designed based on the study of functions and derivatives to analyze the elements of knowledge identified in a mathematics teacher who teaches pre-university students. Specifically, the activity involves conducting a complete study of the sigmoid function and answering a series of reflection and reasoning questions. A sigmoid function is a common S-shaped curve (**Figure 1**) defined by $f(x) = \frac{1}{1+e^{-x}}$, which has applications in various fields such as artificial intelligence, logistic regression, physics, biology, medicine, economics, finance, and signal processing, among others (Kyurkchiev & Markov, 2015).

Due to its mathematical properties and its varied applications in different social, natural, scientific, and technological contexts, we believe that the sigmoid function is a valid function for characterizing the specialized knowledge of mathematics teachers from an integrated and interdisciplinary perspective. Based on the MTSK model, we pose the following research questions:

1. What elements of knowledge, both mathematical and didactic, are identified in the mathematics teacher when conducting a complete analysis of the sigmoid function in an 11^{th} grade course?
2. Are any conceptual, practical, or didactic limitations detected in the teacher's knowledge when implementing the activity in the classroom?

THEORETICAL FRAMEWORK

Since Verret (1975) introduced the concept of didactics as a process of knowledge transfer, various ways of understanding and approaching this concept have been proposed within the academic context, considering the specificity and specialization of the knowledge that is intended to be taught. Shulman (1986) sought to address this process of didactic transposition from the perspective of the discipline itself, analyzing the structure of professional knowledge required by any profession. Shulman (1986) emphasized that it is not enough for teachers to simply know the concepts they are going to teach; it is also important for them to understand the organizational structure and the formality in transmitting these concepts. In this sense, Shulman (1987) highlighted PCK as knowledge that is exclusively the responsibility of teachers, making the discipline's knowledge more comprehensible to students.

From the perspective of mathematics education, many researchers have adopted Shulman's (1986) PCK and reformulated it within their respective theoretical studies to emphasize the nature of mathematical knowledge (MK). For example, Ponte (1994) explored knowledge, beliefs, conceptions, and practices in problem-solving, paying special attention to the teacher's specific knowledge for teaching mathematics. Ball et al. (2008), in turn, managed to characterize Shulman's (1986) pedagogical knowledge within the mathematical context by proposing an analytical model to structure and operationalize the knowledge required by mathematics teachers: the MKT model.

Others, such as Godino (2009), introduced didactic suitability as a tool for mathematics teachers to organize mathematical instructional processes and seek effective classroom intervention. Finally, Carrillo et al. (2013) addressed the specialized knowledge of mathematics teachers through a model that serves as a tool to help organize, explore, and understand the knowledge of mathematics teachers, which, as a whole, makes sense only to them (Sosa et al., 2016). This model is theoretical and aims to analyze, identify, and characterize the knowledge that makes a mathematics teacher a specialist, distinguishing them from other professionals in the field (Escudero-Ávila et al., 2015).

The MTSK model is based on the work of Shulman (1986) and Ball et al. (2008). On one hand, it maintains both the teacher's own subject knowledge, PCK, and Shulman's (1986) curricular knowledge, referring to how teachers manage their classes, organize and structure activities, plan teaching units, ask questions, or even assess students' understanding. On the other hand, it also incorporates the contributions to the conceptualization of the mathematics teacher's knowledge developed by Ball et al. (2008) in the MKT model. However, Carrillo et al. (2013) considered that the MKT model had shortcomings, particularly concerning the exclusivity of the knowledge a mathematics teacher must possess. As a result, they aimed to clarify and reformulate the existing analytical models by creating a more comprehensive one that would group and organize the knowledge exclusive to mathematics teachers.

In this way, the MTSK model is divided into two large domains of knowledge: MK, which deals with aspects inherent to the knowledge derived from the teacher's training and experience in the field of mathematics, and PCK, which, as previously indicated, refers to knowledge about didactics and pedagogy closely linked to MK. These two domains "operate together, inform, and guide the decisions and actions that teachers must take in the course of their teaching" (Carrillo-Yáñez et al., 2018, p. 18).

There is a third domain, which interacts with each of the subdomains of MK and PCK, that of the mathematics teacher's beliefs and conceptions. This domain is further divided into two subdomains: Beliefs about mathematics and Beliefs about mathematics teaching and learning. While the first is related to the knowledge elements of MK, the second connects with the knowledge elements of PCK. However, both refer to the teacher's views or interests regarding the mathematical content taught and the methods used to teach it. As Aguilar-González et al. (2022) point out, "studying them together allows us to have a more complete image of the object to be analyzed" (p. 111).

Table 1, adapted from Delgado-Rebolledo and Espinoza-Vásquez (2021), outlines the structure of the MTSK model and specifies the description of its elements based on the approaches presented by Carrillo-Yáñez et al. (2018). On the other hand, **Figure 2** illustrates the model's structure.

Table 1. Structure of the MTSK model and description of the knowledge elements

Domain	Subdomain	Description of knowledge
MK	Knowledge of topics (KoT)	Mathematical content knowledge features specific to a given concept and its meaning, such as context, representational registers, definitions, properties, or procedures.
	Knowledge of the structure of mathematics (KSM)	Mathematical relations that teachers carry out between different contents, either from the level they are teaching or from other more advanced educational levels (complexification or simplification of a concept).
	Knowledge of practices in mathematics (KPM)	Ways of creating, communicating and writing mathematics (syntax, rigor, analysis, deduction, organization, ...) as well as knowing how to define, axiomatize, prove, disprove, etc.
PCK	Knowledge of mathematics teaching (KMT)	Use of resources (material and technological) and identification of theories, techniques or teaching strategies that enhance mathematical flexibility
	Knowledge of features of learning mathematics (KFLM)	Characteristics and learning theories constitute student interaction with mathematical content: strengths, limitations, typical errors associated with learning; forms of student interaction with the content of habitual language used; and student interests and expectations in approaching the content.
	Knowledge of mathematics learning standards (KMLS)	Conceptual, procedural and mathematical reasoning skills according to curricular objectives: location and context of the content, sequencing of the concepts and topics involved, objectives and associated learning standards, appropriateness of the content to the level, etc.
Beliefs	Beliefs on maths	Emotional impact, affective and cognitive needs associated with the MK domain.
	Beliefs on maths teaching and learning	Emotional impact, affective and cognitive needs associated with the PCK domain.

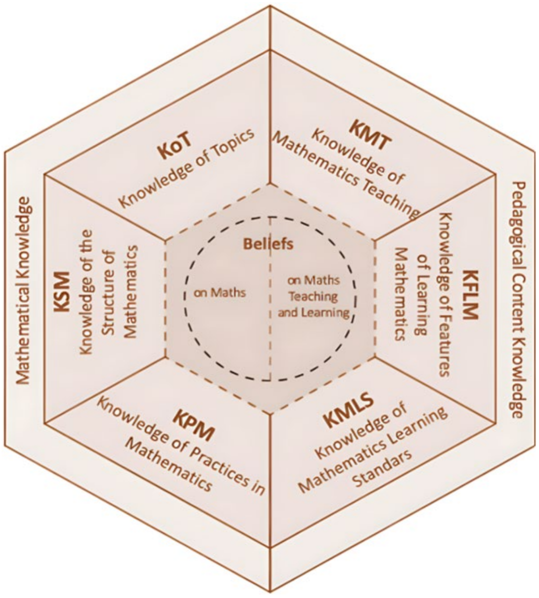

Figure 2. Schematic of the MTSK model (taken from Meléndez-Cruz et al., 2023, originally adapted from Carrillo-Yáñez et al., 2018).

Recent advances in research on the MTSK model have emphasized the evolving nature of the framework and its implications for both theoretical and practical domains of mathematics education. For instance, de Gamboa et al. (2022) proposed the explicit integration of mathematical connections as a central element within the domain of PCK. This refinement responds to the growing need to address the interplay between concepts, procedures, and representational registers in classroom practice, aiming to overcome previous limitations in how specialized knowledge was categorized (Delgado et al., 2022; Flores-Medrano, 2022).

Complementing this perspective, Juárez-Ruiz et al. (2025) introduced a conceptual framework based on Piaget's (1975) schemas theory to analyze the complexity of the internal connections teachers make between knowledge components. Their case study revealed different types of cognitive structures that influence how future teachers integrate elements of the MTSK model, thus offering new insights into the mental organization and variability of specialized knowledge.

In parallel, another relevant development is the theoretical articulation between the MTSK model and other conceptual frameworks, such as mathematical working space (MWS). The work of Flores-Medrano et al. (2016) suggests that combining these two perspectives enables a more holistic analysis of mathematics teaching, as it allows the interplay between teacher decision-making and the epistemological-cognitive dynamics of student learning to be more fully addressed. This synergy offers a multidimensional view of teaching, where both the structure of professional knowledge and the didactic construction of mathematical meaning are foregrounded.

Ultimately, both frameworks converge in their commitment to analyzing knowledge in action, that is, how teachers mobilize their expertise in concrete educational settings. MTSK organizes and categorizes the professional knowledge specific to mathematics teachers, while MWS concentrates on how MK is generated, represented, and validated in the classroom setting. The notion of didactic systems within MWS, understood as functional mechanisms that orient pedagogical choices, strengthens the explanatory power of MTSK by embedding teacher actions within broader epistemological configurations (Panqueban et al., 2024). This integration, explored in recent studies like Espinoza-Vásquez et al. (2025), offers a richer and more contextualized understanding of how specialized knowledge operates in practice.

Beyond these theoretical contributions, several empirical studies have tested the applicability and boundaries of the model, particularly within secondary education settings (e.g., Advíncula et al., 2021; Carrillo et al., 2017; Climent et al., 2021; Espinoza-Vásquez et al., 2016; Montes & Carrillo, 2017; Padilla et al., 2023; Sosa et al., 2015, 2016; Zakaryan & Ribeiro, 2018; Zakaryan & Sosa, 2021). Using qualitative methodologies and focused case study designs, these studies delve into how educators approach mathematical tasks, choose and integrate examples, address student errors, incorporate digital tools like GeoGebra, and connect conceptual content with pedagogical strategies. Covering topics such as geometry, algebra, and functions across multiple educational levels and teaching environments, this body of work illustrates the versatility and relevance of the MTSK model for analyzing instructional practice. Moreover, the findings reinforce the model's usefulness in shaping both pre-service teacher training and ongoing professional development.

METHODOLOGY

Approach

The research adopts a qualitative approach within an interpretive paradigm (Bassey, 2003; González-Monteagudo, 2001) since the objective is to analyze, understand, and interpret the nature of the specialized knowledge of the mathematics teacher when applying an interdisciplinary activity in the classroom. Additionally, an instrumental case study is conducted, as Stake (2007) suggests, to delve into the understanding of a specific topic. In this case, the information provided by the research participants will be used to carry out an abstraction process that will allow for the collection of sufficient data to identify the knowledge the mathematics teacher applies in their teaching practice.

Context

The activity took place in an 11th grade mathematics class at a public school in Madrid. The teacher, whom we will refer to as Évariste, in honor of the mathematician Évariste Galois, holds a degree in mathematics and has five years of experience teaching mathematics in secondary education, from 9th to 12th grade. After completing his studies in mathematics, he earned a master's degree in teacher training with a specialization in mathematics and has since combined his teaching duties with attending and participating in educational conferences and leading numerous didactics workshops for in-training mathematics teachers. Given his academic background in mathematics, his training in mathematics education, and his active involvement in teacher development programs, Évariste was considered to embody the specialized knowledge and reflective practice essential to this study. His openness to collaboration and prior experience with interdisciplinary teaching were additional factors that supported his inclusion in the study. This choice aimed to ensure that the case would provide rich and meaningful insights aligned with the objectives of the research.

The intervention took place in two sessions of fifty minutes each, immediately following Évariste's completion of the topic on applications of derivatives. It is assumed that all the mathematical concepts in the

Table 2. Sequence of exercises in the activity

Sequence		Activity
Introduction	Introduction of the activity	Presentation and objectives Introduction of the sigmoid function
Development	Analysis of the sigmoid function (properties and characteristics): $f(x) = \frac{1}{1+e^{-x}}$	Domain Image or range Intersection points with the axes Symmetry Asymptotes Monotonicity (increasing and decreasing intervals) Critical points (relative maxima and minima) Curvature (concavity and convexity intervals) Inflection points
Conclusion	Representation of the sigmoid function	To make an approximate sketch of the function considering all the characteristic elements calculated in the development.
	Reflection and reasoning questions	Do you remember any other function with a similar shape? Explain the meaning of the parameter r in the function: $f_r(x) = \frac{1}{1+e^{-rx}}$, $\forall\, r \in (0, \infty)$.

calculus area that preceded derivatives and that will be applied in the activity are already known, such as the resolution of equations and inequalities (exponential and logarithmic), functions and their properties, operations with functions, or the calculation of limits of functions at infinity. A total of fifteen students voluntarily participated in the sessions, all of whom were science majors.

Prior to the intervention, all participating students were fully informed about the purpose, procedures, and voluntary nature of the study through a participant information sheet. Additionally, the teacher gave his consent to participate in the research, and measures were implemented to protect the anonymity and confidentiality of all participants, including the use of pseudonyms and secure data handling. Furthermore, approval was obtained from the school administration to conduct the study, thereby ensuring compliance with ethical standards related to educational research.

Data Collection and Evaluation Instruments

Classroom observations and field notes were used for data collection, both a priori and a posteriori (Kagan, 1990). The classroom observations followed the non-participant direct observation method (Cohen & Manion, 2002), meaning the researcher does not interact or get involved in the teacher's work but instead remains in the background, observing and collecting as much relevant information as possible to address the research questions.

Regarding the field notes, a written descriptive and reflective record of the observations made, experiences lived, and perceptions collected during the intervention was chosen (Lofland & Lofland, 1984). We believe that, in this case, they are a fundamental resource for understanding the context, nuances, and complexities of the research environment. At the end of the intervention, the different perspectives and personal reflections of the researchers were included. This allowed us to characterize and analyze, based on the MTSK model, the different elements of knowledge identified during the Évariste intervention.

Likewise, researcher triangulation was encouraged through joint sessions for discussing and interpreting the data, which enriched the understanding of the observed phenomena and helped minimize potential individual biases. By following this analytical framework, the study aims to establish a clear and rigorous link between the empirical data and theoretical constructs, thereby ensuring transparency and depth in the interpretative process.

Activity Design

The activity consists of three clearly differentiated parts:

(a) the introduction, where the teacher introduces the sigmoid function,

(b) the development, where a complete study of the function is conducted, and

(c) the conclusion, where the function is graphed, and some reflection and reasoning questions are posed.

Specifically, **Table 2** shows the exercises that make up the activity.

RESULTS

From now on, we will indicate in brackets the subdomains from which the knowledge elements identified during Évariste's intervention come. It should be noted that the knowledge elements corresponding to the KMLS subdomain and the beliefs domain of the MTSK model are not the focus of this research.

Results Obtained in the Introduction of the Activity

The first indicator of knowledge that emerges directly from the exercises included in the approach is the necessity of knowing all the characteristic properties required to study a function and represent it adequately. However, it should be noted at this point that the way the sigmoid function is introduced is crucial for understanding its structure.

Évariste writes the function on the board, and the first thing he asks his students is, "*What type of function is this?*" Some students respond that "*it is an exponential function because it has the number e raised to something with x*". Évariste agrees but emphasizes that it is important to qualify this answer, which is why he corrects them by saying, "*It is exponential because the function contains a real number raised to another function that depends on the study variable x*" [KoT, KPM]. He comments that "*a function is not only exponential because the number e appears raised to a function with x, but that the base can be any real number, and the exponent can be any function that depends on x*" [KoT, KPM].

At this point, Évariste contributes examples of other exponential functions [KMT, KPM], referring to the topic of exponential and logarithmic functions and the topic of exponential equations that they had already studied [KSM]. He adds, "*Exponential functions are extremely important in many fields of study, such as biology, economics, or physics, due to their ability to model processes of growth and decay in a population*" [KoT, KPM]. Here, Évariste contributes the knowledge element of relating the concept of exponential functions with other areas of knowledge outside of mathematics [KSM]. Additionally, Évariste reminds his students of the shape of exponential functions by drawing examples on the board [KoT, KPM] and slightly highlighting some of their main properties [KoT].

Another important point to highlight is the question asked by a student regarding the symbols used: "*What does the inverted A mean? I don't remember*" [KFLM]. Évariste had written the function on the board as follows [KPM]: $f(x) = \frac{1}{1+e^{-x}}$, $\forall\, x \in \mathbb{R}$, without stopping to explain the symbols used. Although this symbology is the most common for him, some students are still not familiar with it [KFLM]. Évariste clarified, "*It means for every value of x belonging to the real numbers*" [KoT]. With this clarification, Évariste reminded the students that ∈ is the symbol of belonging, anticipating that they might ask about it [KFLM].

Later, Évariste asks, "*Is the function only exponential?*" One of his students responds, "*It is also rational because it has a numerator and a denominator*". After asking the rest of the students if they agreed with their partner and receiving a positive response, Évariste reminds them, "*A rational function is one that can be expressed as the quotient of two polynomials*" [KoT] and asks the question again: "*Knowing this, is this function rational?*". One student responds, "*Then no, because the denominator is not a polynomial*". Évariste nods and comments that what can be said is that "*the sigmoid function is a composition of a rational function and an exponential function because it can be expressed in the form*" and he writes on the board: $f(x) = (g \circ h)(x) = g(h(x))$, where $g(x) = \frac{1}{1+x}$ and $h(x) = e^{-x}$, or taking $g(x) = \frac{1}{x}$ and $h(x) = 1 + e^{-x}, \forall\, c \in \mathbb{R}$ [KPM, KMT]. In this way, he introduces the concept of function composition, although it is not necessary for the development of the exercise [KoT, KSM, KFLM].

Finally, Évariste asks if they think there is another way to express the same function without changing its properties [KFLM], but no student answers. Sophie insists, "*What if we transform the exponent of the number e to positive? How would the function vary?*". A student responds, "*By putting a 1 in the numerator and moving the number e with a positive exponent to the denominator*". Évariste continues explaining: "*Then the function is transformed to* (writes on the board): $f(x) = \frac{1}{1+\frac{1}{e^x}} = \frac{1}{\frac{e^x+1}{e^x}} = \frac{e^x}{e^x+1}, \forall\, x \in \mathbb{R}$".

In this way, Évariste proposes another way of expressing the function [KoT, KMT] and, in the process, uses basic concepts of powers and operations with fractions [KSM].

As can be seen, the sequence followed by Évariste to introduce the sigmoid function is well-structured and organized. It uses mathematical symbols, its explanation has a solid mathematical foundation, and it serves to reflect on some mathematical concepts before starting the development part of the activity [KPM, KSM, KMT]. Likewise, he uses examples and numerical and graphic records that help introduce the concept of study [KoT, KMT] and during the explanation he interacts with the students using a formal but understandable vocabulary for them [KFLM].

Results Obtained in the Development of the Activity

We now focus on presenting the results derived from the students' participation and the teacher's interaction during the development of the activity. Évariste considered that the best way to guide the activity was to study each property of the function point by point, while also interacting with the students by asking them questions about the content. This approach allowed him to identify both the students' strengths and weaknesses, detect possible errors, and anticipate them whenever possible [KFLM].

In the first section, Évariste first introduces the concepts of domain and intercepts with the axes. He notices that some students have difficulty solving the equation $e^{-x} = -1$, which is a priori easy to solve. In this regard, the following dialogue between Évariste and some students deserves evaluation:

Évariste: Remember from the topic of exponential equations what needs to be done to solve an unknown.

Student 1: We had to make a variable change.

Évariste: Okay, and what change would you make?

Student 1: The only one that can be done is $t = e^{-x}$ or $t = e^{x}$, I don't care.

Évariste: Okay, so what would you have left?

Student 1: $t = -1$, but when I undo the variable change, I go back to the beginning.

Évariste: Okay, this means that the equation does not allow for a variable change, so we need to look for another alternative.

Student 2: You have to apply logarithms on both sides to remove the e.

Évariste: Very well, as your partner says, we can solve the equation directly by applying logarithms on both sides of the equation.

Student 1: Okay, I get it. Then I would be left with $-x = \mathrm{Ln}(-1)$, but since the natural logarithm does not exist for negative numbers, it has no solution.

Évariste: Very good. So, what can we say about the domain?

Student 1 and student 2: It's the set of real numbers.

Student 3: But wasn't this already known from the beginning because it was an exponential function?

Évariste: Keep in mind that the exponential function is in the denominator, so if there is any value of x that cancels the denominator, it wouldn't be part of the domain. You need to be careful with that. For instance, imagine that the denominator was $1 - e^{-x}$. Then, $e^{-x} = -1$, and its solution would be $x = 0$. The domain of the function would be the set of real numbers except zero, which, in this case, we cannot include. In any case, realize that we don't need to carry out all these calculations if we reason properly. We can say that it is impossible for e^{-x} to equal –1, because if we replace x with any real number, the result will always be positive, and never negative.

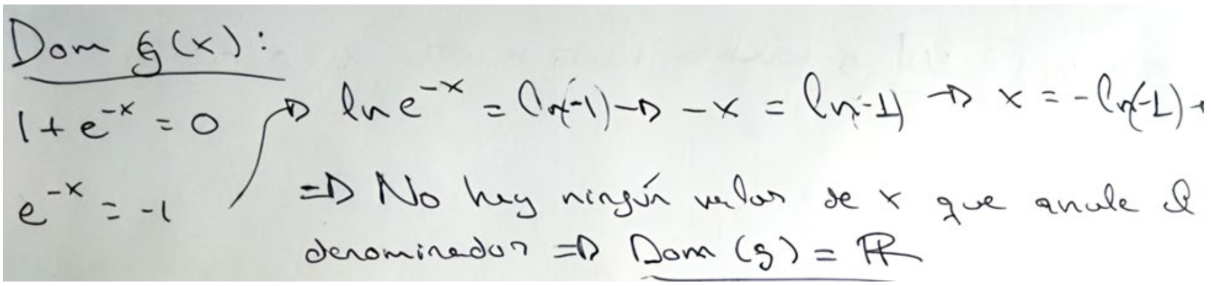

Figure 3. Calculation of the sigmoid function domain by one of the students: The student equates the denominator to zero and ultimately states that there are no values of *x* that make the equation undefined, concluding that the domain is all real numbers (Image taken from a student's written solution, used with informed consent as part of the research)

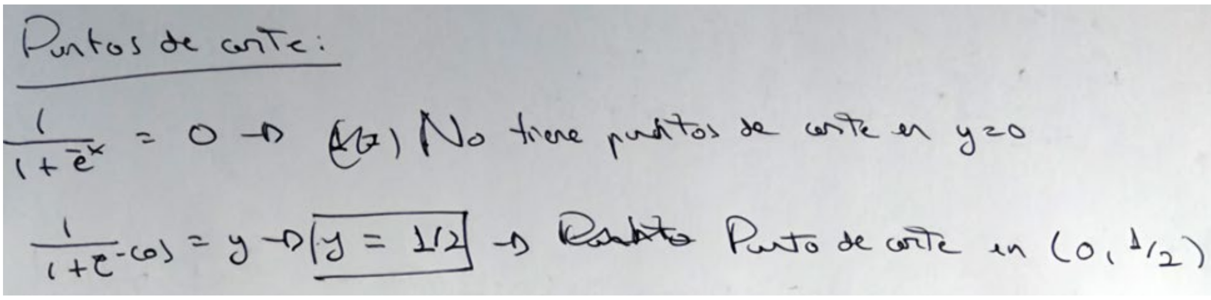

Figure 4. Calculation of the intersection points with the axes by one of the students: The student solves the equation $f(x) = 0$ and, upon finding no real solutions, concludes that the function has no x-intercepts & Additionally, by evaluating the function at $x = 0$, the student determines that the y-intercept is at the point $\left(0, \frac{1}{2}\right)$ (Image taken from a student's written solution, used with informed consent as part of the research)

From this sequence, we detect several elements of knowledge. We can identify in Évariste knowledge of domain calculus for rational functions [KoT] that he can relate to solving exponential equations [KSM]. Moreover, he offers other ways to solve the exercise [KMT] and does not limit himself to telling students how to do it but rather guides and orients them in the process, allowing them to realize the mistakes they are making [KMT, KFLM]. He also provides counterexamples to correct a student's misconception [KPM]. **Figure 3** and **Figure 4** show some of the solutions the students provided.

Regarding the calculation of the image, more problems arose among the students because the procedure includes concepts from other topics that needed to be reviewed, such as the calculation of the inverse function and the calculation of inequalities. However, some students solved it without any issues (**Figure 5**). Below, we include the conversation that occurs between Évariste and some students about this concept:

> **Student 1:** Professor, I don't remember how the image of a function was calculated if I don't see it represented. Can we do it later at the end when we perform it?
>
> **Évariste:** No, you have to know how to do it analytically, without relying on representation. Does anyone remember how it's done?
>
> **Student 2:** Yes, you had to find the inverse of the function and calculate its domain.
>
> **Évariste:** Very good. Remember that the image of a function is equivalent to the domain of its inverse function. Therefore, the process to follow here is to first calculate the inverse of the sigmoid function and then calculate its domain. To find the inverse, you first have to swap the variables and then solve for y. Let me write it on the board: $x = \frac{1}{1+e^{-y}}$.

To solve for y, you first have to multiply the denominator by the other side and then isolate the terms depending on y on one side, and those depending on x on the other side. Do you see that the variable we

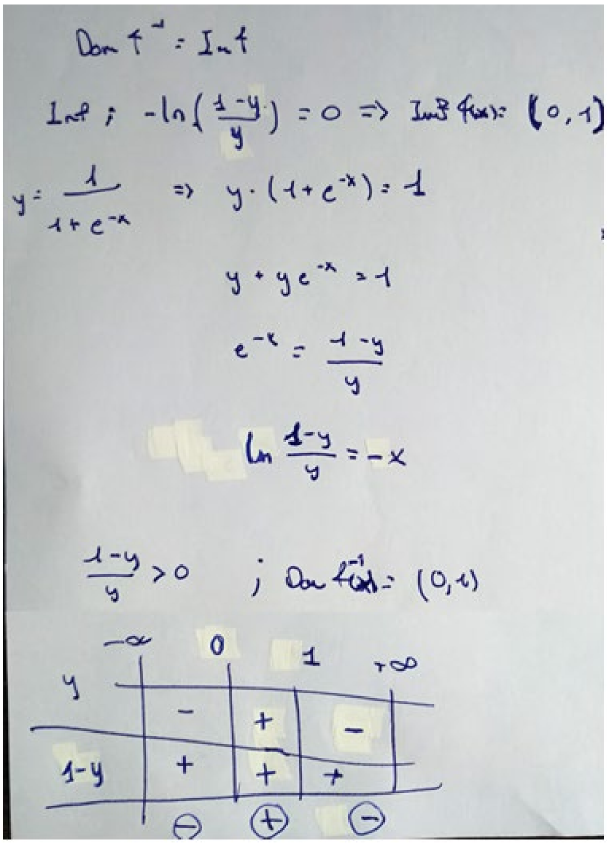

Figure 5. Calculation of the image of the sigmoid function by one of the students: In this case, the student chooses to determine the range analytically by calculating the domain of the inverse function & The student concludes that the range of the function is the interval $(0, 1)$ (Image taken from a student's written solution, used with informed consent as part of the research)

want to solve for is in the denominator of an exponential? This tells us with complete certainty that we must take logarithms on both sides to solve for y.

With these guidelines, all students were able to find the inverse of the function without any problems. It can be seen how Évariste knows the concepts involved in the process of analytical image calculation [KoT] and guides his students by referencing concepts from previous topics, such as the calculation of function inverses and solving exponential equations [KSM]. Aware of the students' limitations at this point, Évariste anticipates the mistakes they might make and provides relevant guidance to avoid them using vocabulary familiar to the students [KFLM].

In general, the biggest problem arose when solving the resulting inequality after calculating the inverse function, as many students admitted they had forgotten how to handle this issue. This suggests that students tend to mechanize the exercises instead of focusing on the procedure, which does not guarantee a deep understanding of mathematical concepts. However, Évariste helped them by reminding them of the steps needed for their calculation, occasionally referring to graphical records [KoT]. Specifically, Évariste identified a serious error in some of his students' resolution of the inequality, which merits attention. To put this in context, we know that the inverse function is $f^{-1}(x) = -\text{Ln}\left(\frac{1-x}{x}\right)$ and to calculate its domain, one must solve the inequality $\frac{1-x}{x} > 0$, since a logarithm cannot take negative values. Students perform the following steps: $1 - x > 0 \Rightarrow -x > -1 \Rightarrow x > 1$, and conclude that the image of the function is the interval $(1, \infty)$. Below is part of the conversation that Évariste had with the group:

> **Évariste:** I see two very serious flaws in this procedure. The first is that the implication $-x > -1 \Rightarrow x > 1$ is not true. What mistake do you think you are making here?

Student 1: Well, I don't see it because since both sides are negative, they cancel each other out, right?

Évariste: According to you, the solution to the inequality is any value of x greater than 1. Take a value greater than 1 and substitute it into the inequality to see if it holds.

Student 1: It's true. If I take $x = 2$, I get $-1/2$, which is not greater than 0.

Évariste: So, what's the flaw in your reasoning?

Student 1: Okay, I see it now. When x is negative, you have to flip the inequality sign. So, I would have $x < 1$.

Évariste: Well, it's a strange explanation, but it works for me. However, this is not the most serious error. Keep in mind that the inequality has both a numerator and a denominator. In your reasoning, you moved the denominator to the other side by multiplying, and it disappeared. But you can't do that because you must remember that the values that cancel the denominator must also be considered, right?

Student 2: Ah! I remember now. We have to set both the numerator and denominator equal to 0 and evaluate the intervals they form. So, from the numerator we get $x = 1$, and from the denominator, $x = 0$. We need to evaluate the intervals "less than 0, between 0 and 1, and greater than 1", keeping the positive ones, right?

Évariste: Very good. But what do you need to do to find out which interval remains?

Student 2: We replace x with a number from each interval and check if the result is positive. If it is positive, it's part of the solution, and if it's negative, we discard it.

Évariste: Perfect. I'm sure you won't make the same mistake next time.

From this extract, we deduce that several elements of knowledge need to be considered. Évariste demonstrates MK regarding the calculation of inequalities [KoT]. The vocabulary he uses in his explanations is not too complex, and he avoids using mathematical expressions that could make the concept harder to understand. Instead, he uses language that is familiar to the students [KoT, KFLM]. Additionally, he helps them remember the procedure through guided questions and is always mindful of the students' limitations, as the error they made is very common [KFLM].

Regarding the study of symmetry, all students agree that the function has neither even nor odd symmetry because the equalities for these types of symmetry do not hold. However, at this point, Évariste explains that this function exhibits a type of symmetry called midpoint symmetry, where the function's shape is symmetrical around a specific point, which turns out to be the inflection point [KoT]. Évariste refers to this property by writing on the board: $f(x) + f(-x) = C$.

He explains that, for these types of functions, the sum of $f(x)$ and $f(-x)$ will always equal a constant C, which turns out to be the difference between the absolute value of the maximum value and the absolute value of the minimum value of the function [KoT, KPM]. Évariste writes on the board: $f(x) + f(-x) = |y_{SUP}| - |y_{INF}|$.

Évariste encourages his students to prove that in the sigmoid function, the equality $f(x) + f(-x) = 1$ holds.

As for the calculation of asymptotes, all students demonstrate the ability to calculate them correctly. A very common mistake at this point is stating that $a^{-\infty}$, for all $a \in \mathbb{R}$. Aware of this potential error, Évariste reminds students before they make it that when calculating a power with a negative exponent, the exponent must first be transformed to a positive value [KoT, KFLM]. In this way, students calculate the asymptotes without problems (**Figure 6**).

Regarding the study of monotonicity, critical points, curvature, and inflection points of the function, Évariste had previously reminded students of the procedure to follow, urging them to use a scheme he had

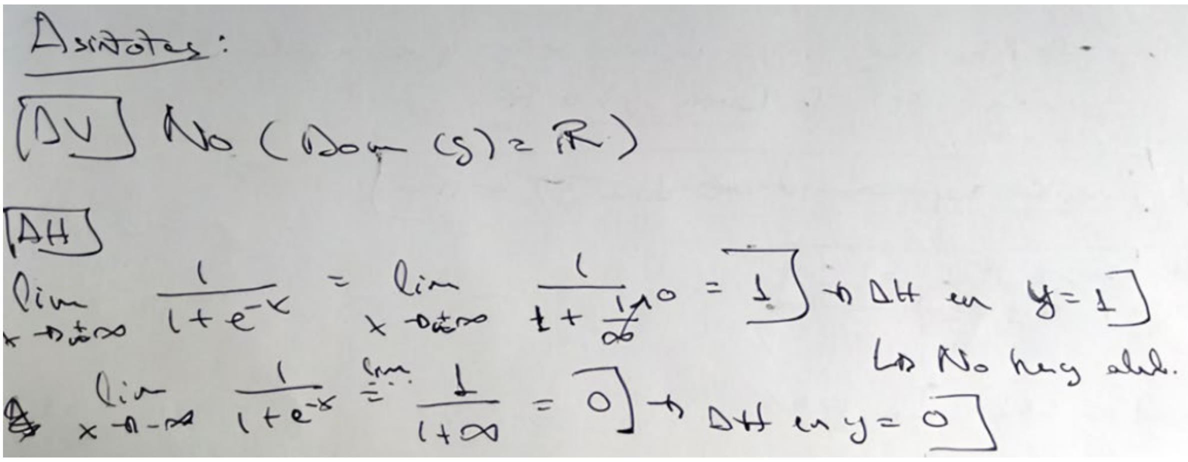

Figure 6. Calculation of the asymptotes of the sigmoid function by one of the students: The student states that there is no vertical asymptote, since the domain is all real numbers (to determine the horizontal asymptotes, the student computes the limits of the function as x approaches $+\infty$ and $-\infty$, concluding that the function has horizontal asymptotes at $y = 0$ and $y = 1$) (Image taken from a student's written solution, used with informed consent as part of the research)

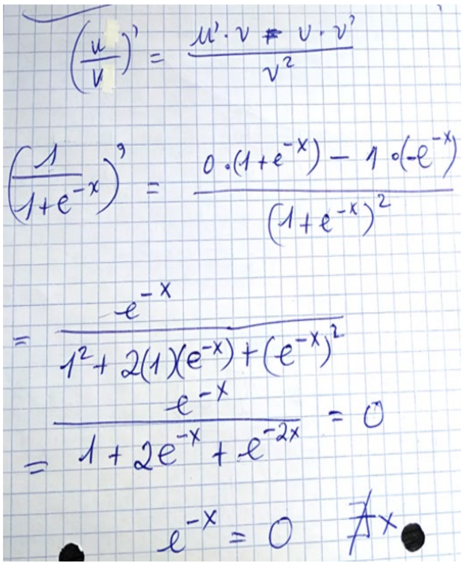

Figure 7. Calculation of the first derivative of the sigmoid function by one of the students: The student chooses to write out the quotient rule, denoting the numerator as u and the denominator as v (the student computes the derivatives separately and then applies the rule & instead of using the formal notation $\frac{d}{dx}$, the student refers to the derivatives using the prime symbol) (Image taken from a student's written solution, used with informed consent as part of the research)

provided [KMT (use of material resource)]. With this scheme, several knowledge elements of the teacher are detected: he knows the procedure to study the properties and characteristic points of a function [KoT], uses graphical records to help understand the process [KoT, KMT], uses appropriate syntax with symbols and mathematical notations, and understands the logical sequence of the procedure [KPM].

We observe that students have no problems calculating the first derivative (**Figure 7**), but they struggle when studying monotonicity and critical points. Below is the conversation that Évariste had with a group of students on this topic:

> **Évariste:** I see that you have expanded the notable identity $(1 + e^{-x})^2$ of the denominator. Do you think this development will help in studying the possible critical points of the function?
>
> **Student 1:** I haven't thought about it. I just did it because it's normal.

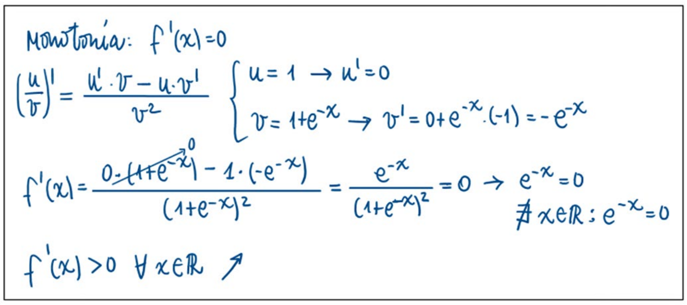

Figure 8. Teacher's correction: Monotonicity and critical points (Image taken from teacher's written solution, used with informed consent as part of the research)

Évariste: The derivative must be equal to 0, and when solving the equation, the denominator will cancel out. So, why perform a calculation that is unnecessary?

Student 1: I get it, but to find the second derivative, we will need to expand it, right?

Évariste: Well, it's not necessary if you know how to differentiate a potential function. But if you don't remember, expanding the identity before differentiating is another option.

Student 2: Teacher, by setting the first derivative equal to 0, I end up with $e^{-x} = 0$, which is an equation that has no solution. So, how do we study monotonicity if we have no values to work with?

Évariste: Remember, we must not only consider the values of x that result from setting the first derivative equal to 0. What other values do we need to consider?

Student 3: Those that are outside the domain. But the domain is all real numbers, so we still don't have points to study monotonicity.

Évariste: Very good. So, we need to evaluate the entire set of real numbers, from negative infinity to positive infinity. We evaluate any value from this set in the derivative. What happens for one value will happen for the entire set, do you understand?

Student 3: So, if I take $x = 0$, which is the simplest, I get that the derivative is $1/2$, which is positive. This means the function increases, but on what interval? From minus infinity to infinity?

Évariste: Very good. If you try real numbers in the derivative, all of them will be positive. Keep in mind that the numerator is always positive, and the denominator is also positive because e^{-x} is always positive for all $x \in \mathbb{R}$. If we take a negative value, for example, $x = -2$, we get e^2, which is positive, and if we take a positive value, for example, $x = 2$, we get $e^{-2} = \frac{1}{e^2}$, which is also positive.

Student 3: So, there aren't any relative extremes, right?

Évariste: Exactly, if the function increases throughout $\mathbb{R}$, then it is impossible for there to be relative maxima or minima. The same would be true if the function decreased throughout $\mathbb{R}$.

Note how, from this piece of conversation, several elements of knowledge are detected in Évariste. First, he identifies basic errors or unnecessary calculations that will not be useful for the desired calculation [KFLM]. He also demonstrates knowledge of how to study the monotonicity of a function when there are no candidates for critical points [KoT]. Furthermore, Évariste is aware of his students' limitations and provides examples and properties to help them better understand the concept [KPM, KMT].

Finally, the study of curvature and inflection points presented more challenges, as calculating the second derivative requires organization and simplification. In this case, Évariste chose to explain this section on the board, demonstrating the process step by step (**Figure 8** and **Figure 9**).

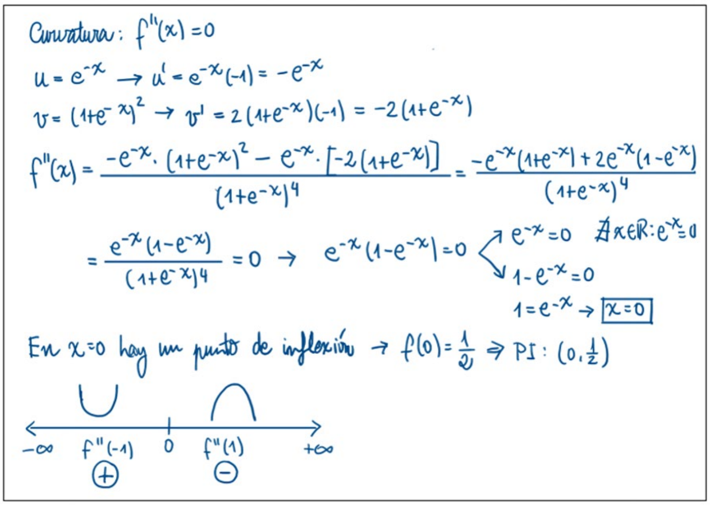

Figure 9. Teacher's correction: Curvature and inflection points (Image taken from teacher's written solution, used with informed consent as part of the research)

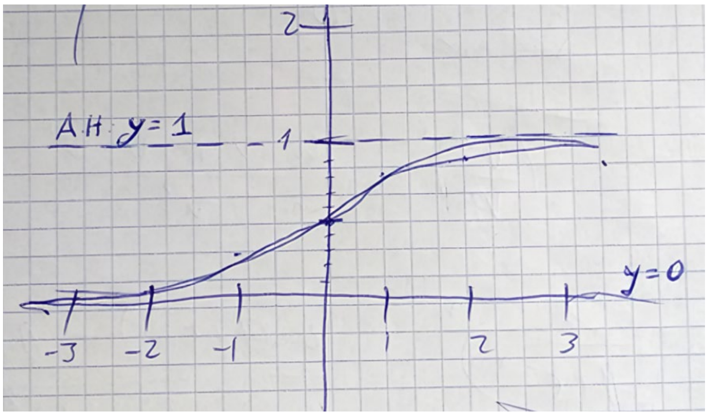

Figure 10. Representation of the sigmoid function by one of the students: The student plots the horizontal asymptotes and the intercept point, which is also identified as the point of inflection (from there, they sketch the graph of the function based on its monotonicity and concavity) (Image taken from a student's written solution, used with informed consent as part of the research)

As can be seen in **Figure 8** and **Figure 9**, Évariste uses a syntax appropriate for the level, and the mathematical procedure is carried out with sufficient rigor. He demonstrates deductive reasoning and follows a logical and orderly sequence in his approach to solving the mathematics of the activity. Specifically, he shows a high degree of mathematical maturity by using symbols and mathematical structures in his work [KPM]. We also observe how he employs heuristic learning techniques that, based on his experience, help better understand the procedure, such as writing the quotient rule for calculating the first and second derivatives in an organized manner (**Figure 8**) or using graphical representations to calculate the curvature of the function (**Figure 9**) [KMT]. In the latter case, note that the third derivative is not applied to verify that there is indeed a inflection point at $x = 0$; this is simply shown through the curvature justification diagram, demonstrating mastery of the KPM knowledge elements.

Results Obtained in the Conclusion of the Activity

As for the representation of the function, Évariste advises students to write down all the data they have, as this will help them reason about its form [KPM, KMT]. While many students make rough sketches of the function, others opt to create a table of values to make it as precise as possible (**Figure 10**).

Finally, let us analyze the knowledge elements identified in the reflection and reasoning questions. Regarding the reflection question: "Do you remember any other function with a similar shape?" the following dialogue between Évariste and his students is excerpted:

Évariste: Do you remember any function from the function topic that has the same form? **Students**: The arctangent.

Évariste: Very good. Do you remember what properties it had?

Student 1: (Students review notes from the function topic and begin listing some properties) The domain is all real numbers, the image goes from $-\pi/2$ to $\pi/2$, ...

Évariste: (Interrupts student 1) Be careful, it's not said that the image goes from $-\pi/2$ to $\pi/2$, but rather that the image is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, or that the image set is all real numbers between $-\pi/2$ and $\pi/2$, neither of which are included.

Student 1: It also passes through the origin and has horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.

Évariste: Very good, any other properties?

Student 3: It is also continuous and increasing.

Évariste: (Évariste corrects and emphasizes) Continuous and increasing throughout its domain or continuous and increasing over $\mathbb{R}$. What can you tell me about curvature and inflection points?

Student 4: It is concave from ∞ to 0 and convex from 0 to ∞ and has a inflection point at 0.

Évariste: Be careful, inflection point at (0,0), as points have both an x and y component. What can we say about symmetry? Is it symmetric about the inflection point, as in the case of the sigmoid function?

Student 2: Yes, because it has the same shape.

Évariste: So, is it also true that $f(x) + f(-x) = 1$?

Student 2: No, in this case, the sum would be 0.

Évariste: Very good. Notice that the image of the sigmoid function was (0,1), and that of the arctangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, if we recall the property I mentioned, it will give us, in this case (writes on the board): $f(x) + f(-x) = \left|\frac{\pi}{2}\right| - \left|-\frac{\pi}{2}\right| = \frac{\pi}{2} - \frac{\pi}{2} = 0$.

In this dialogue, Évariste demonstrates his knowledge of trigonometric functions and their main properties [KoT, KSM] and interacts with the students to ensure they solve the exercise themselves, assessing whether they remember the concept being studied [KFLM]. He also corrects students when they use incorrect mathematical language [KPM, KFLM].

Furthermore, Évariste decided to use GeoGebra to represent the function $f(x) = \frac{x}{\sqrt{1+x^2}}$, whose form is also similar to the sigmoid function and asked his students again about its properties [KFLM]. Once again, Évariste demonstrates his knowledge of this type of function by proposing a rational function, which is generally more familiar to students than a trigonometric function [KoT, KSM, KMT].

Regarding the reasoning question about the meaning of the parameter r in the function: $f(x) = \frac{1}{1-e^{-rx}}$, $\forall\, r \in (0, \infty)$, Évariste, aware of the difficulty this posed for his students, helped them by saying: "*We know what shape the function has for $r = 1$. Well, think about what would happen if we halved that value, for example. Would the curvature of the function be more or less pronounced?*"

This is a good strategy for helping students understand how the [KMT] function changes, and at the same time, Évariste demonstrates his knowledge of the mathematical concepts involved in the [KoT] exercise. Specifically, we extract the following dialogue:

Évariste: Realize that if we decrease the value of r to 0, what would the form of the function be like?

Student: It would be completely horizontal because if $r = 0$ is substituted into the function, we get $f(x) = 1/2$, which is a constant function.

Évariste: Very good. So, this means that as r tends to 0, the curvature becomes less and less pronounced. Do you understand this? So, what would happen if r gets larger and larger?

Student: The opposite, the curvature will become more and more pronounced.

Évariste: So, what meaning can we give to the parameter r? (No student responds). It would mean that the function grows faster for large values of r and grows more slowly as r approaches 0, right?

Évariste: Be careful if we give negative values to the parameter, because in that case, the function would be decreasing throughout $\mathbb{R}$.

We can observe that, although he helps his students understand the phenomenon by asking them questions, he doesn't fully achieve it. Therefore, the strategy he followed was to directly represent the function with GeoGebra using a slider for r [KMT (use of technological resource)]. This way, the students understood the phenomenon with less effort.

DISCUSSION

Next, each of the subdomains of the MTSK model is analyzed based on the results obtained, with the purpose of answering the research questions posed in the introduction.

Regarding the KoT, we agree with Vasco and Moriel (2022) that the mathematics teacher must know and understand the content of the mathematics they teach, which includes definitions, properties, procedures, and representation registers (Sosa et al., 2016). As seen in the analysis of the results, Évariste demonstrates his knowledge of the properties and foundations of the mathematical objects involved in the activity, states and understands the elements that make these properties definable, refers to conventional formulas and algorithms, and uses alternative procedures to address the content (Pérez-Montilla & Cardeñoso, 2023). Likewise, he makes use of graphic, numerical, and verbal representation registers (D'Amore, 2004; Godino, 2010a), using representation schemes and appropriate notation in his corrections, as well as employing suitable vocabulary in his classroom interventions.

On the other hand, as Godino (2002) points out, it is essential that the mathematics teacher understands the background of everything they teach to transmit it adequately to their students. This is why it is not enough for teachers to know the what and why of the content they teach; it is also important to explore its epistemological basis and real-life applications. In this sense, it is crucial that the mathematics teacher not only knows the applications of a topic within mathematics itself but also knows how to contextualize the concept and interrelate it with content from other disciplines (Godino, 2010b). We identified this interdisciplinary approach in Évariste, who provided examples illustrating this connection and explaining the phenomenon under study.

As for the KSM, we agree with Flores-Medrano (2022) and Carrillo-Yáñez et al. (2018) that it is essential for the teacher to be aware of the system of connections that articulates MK. In other words, they must be able to connect the most advanced concepts with the most elementary ones and understand the content they teach in depth. In Évariste's case study, there is significant evidence that demonstrates his use of interconceptual connections throughout the intervention, as he is knowledgeable about the mathematical construction process required for the complete analysis of a function.

Concerning the KPM, it is essential to know how a mathematical concept or procedure is defined, argued, or demonstrated, as well as how to transmit that knowledge using appropriate mathematical language and formal syntax (Delgado et al., 2022). In this case, Évariste demonstrates sufficient ease in practice, as his mathematical jargon is broad, his writing is formal and precise, and he uses mathematical symbols that allow him to express or summarize ideas clearly. Furthermore, he uses justification schemes and heuristics that simplify or help to better understand parts of the activity (Dólera-Almaida & Sánchez-Jiménez, 2024). As for presenting the content, he follows a logical and orderly sequence as he advances in the study of the properties of the function, planning and organizing the relationship of the content.

Regarding the subdomains of PCK, we were able to identify in Évariste the main characteristics that a good mathematics teacher must have to perform quality teaching and learning. In general, he demonstrates knowledge of how his students think and interact when approaching the task and is aware of both their abilities and limitations. On numerous occasions, he anticipates the errors they might make, providing support during the process of understanding the activity and effectively guiding them in the learning process.

As for the KMT, as Sosa and Reyes (2022) point out, it is essential that the mathematics teacher understands teaching theories based on their own experience, reflective practice, and research, as well as the impact of the material and technological resources available to address the content. The teacher should also understand the different ways of proceeding when solving an activity (mathematical flexibility). During the intervention, we identified many knowledge indicators in Évariste, arising from the analysis of his use of examples, resources, guidelines, and aids he provides to his students. In line with the research by Sosa et al. (2016) on this subdomain, Évariste demonstrates:

(a) an understanding of the potential of contextual examples to create a meaningful environment and serve as tools to highlight or emphasize unique aspects of the mathematical content to be transmitted,

(b) the ability to choose resources that help students find meaning in the concept being taught, and

(c) the ability to know the type of help to offer students and the strategies to follow in difficult situations.

On the other hand, regarding KFLM, it is essential that the mathematics teacher understands how their students think when facing an apparently complex activity (Escudero-Ávila, 2022). Similarly, teachers must also be able to identify potential difficulties in the task before implementing it, as this will help them predict where students might make mistakes and prevent those errors (Muñoz-Catalán et al., 2015). Based on the knowledge indicators identified by Carrillo-Yáñez et al. (2018) within KFLM, we highlight Évariste's interaction with students to resolve their doubts and assist them in the resolution process, as well as his analytical and deductive capacity to identify which parts of the activity need reinforcement.

Finally, no practical or didactic limitations have been observed in the teacher's knowledge during the intervention, as it is precisely the knowledge indicators of the KPM, KSM, and KFLM subdomains that have been most reinforced. No clear conceptual limitations have been noted, as knowledge of mathematical content is evident. However, some context might be missing in the introductory part of the activity. Évariste is very involved with the mathematical structure of the sigmoid function but does not focus as much on explaining its origin and why it arose, making it necessary to reinforce the epistemological aspects of the content that fall under the KoT. We believe that it is important for teachers to carry out prior research to properly contextualize the activity before implementing it.

CONCLUSIONS

We agree with Carrillo-Yáñez et al. (2018) that for a mathematics teacher's knowledge to be specialized, it is not enough for teachers to simply know the content they teach; they must also understand why they teach it and how to effectively transmit it to their students. In this sense, we can say that the knowledge identified in Évariste during the development of the activity is specialized, as the elements of knowledge involved in the subdomains of the MTSK model were observed, to varying degrees. In fact, although we aimed to show the reader a sample of the model's potential in terms of the functioning of the subdomains individually, the analysis of Évariste's mathematical and didactic knowledge has also allowed us to identify the integrated nature of these knowledge elements, especially when the focus of the activity is interdisciplinary.

We find it important to highlight Évariste's teaching work when addressing a priori complex concepts for students, especially when these involve new content for them, such as in this case, the concept of the derivative and its applications. Conducting a comprehensive study of a function is a somewhat complex task since concepts from various topics in different areas are interrelated, such as set theory, intervals, equations, inequalities, limits, etc. The complexity is heightened when the function is somewhat unusual and not the typical one that students encounter in textbooks. For this reason, it is crucial for teachers to have internalized the elements of MK (those outlined in the KoT, KSM, and KPM) and know how to relate them to those of pedagogical knowledge related to teaching and learning (those in the KMP and KFLM).

The activity was designed with the goal of showing students the broad applicability of mathematics beyond the classroom. The sigmoid function was chosen as the central topic because it is a function used in various fields to study the behavior of certain phenomena. In this regard, we want to highlight two important aspects that we felt were missing in Évariste's case study: contextualizing the function and providing examples (and generalizations) it for real-world cases. We believe that, although Évariste relates the function to certain social and scientific behaviors during the activity, it would have been valuable for the development of the lesson if he had emphasized the importance of this function in real life. This does not imply that Évariste lacks this knowledge; rather, it suggests that he prefers to prioritize the mathematical content over the explanation of the phenomenon. However, when carrying out such an activity, it is essential for teachers to conduct research beforehand to avoid potential conceptual limitations in their knowledge of the topics.

In this regard, we believe teachers must understand the origin of the concept they wish to transmit, know how and why it emerged, who developed it, its historical and cultural context, and the different interpretations and approaches associated with it. Of course, this brings pedagogical implications that must be considered, as teachers must also know how to introduce this information effectively, adapting to the intellectual and emotional needs of the students. In doing so, teachers will promote a deeper and more critical understanding of the topic they wish to address (Organization for Economic Cooperation and Development [OECD], 2023).

On another note, it should be mentioned that the KMLS subdomain of the PCK was not analyzed in this research because, as it is a specific case implemented at a specific time, knowledge about curricular aspects such as the level of application, temporality, sequencing of concepts, key and specific competencies, or learning standards is assumed to be known. The knowledge elements in the Beliefs domain of the MTSK model were also not analyzed, as this requires collecting more specific data on the teacher's prior experiences and personal reflections regarding what it has meant for them to develop the activity with their students. However, we recognize the value of studying these elements in future research, as this domain is worth exploring, especially when applying interdisciplinary activities like the one conducted in this research.

Finally, we consider that the research contributes to the goal of further building the specialization of this knowledge to create new learning opportunities. It also makes clear that, while implementing interdisciplinary activities requires more in-depth analysis, they can be much more enriching for students. Of course, this assumes systemic, formal, and integrated knowledge of the topics and mathematical practice on the part of teachers, whose knowledge elements must be studied in greater detail. For this reason, it is advisable to continue exploring this topic to develop a more detailed understanding of the MTSK.

Contribution to the Field

This study provides the field of mathematics education with a detailed analysis of the specialized knowledge a mathematics teacher employs when conducting an interdisciplinary activity centered on the sigmoid function. By applying the MTSK model, the research helps us better understand how mathematical and pedagogical knowledge come together in real classroom settings, clarifying the relationship between content mastery and teaching strategies. The findings highlight the importance of epistemological contextualization in teaching complex mathematical concepts, suggesting that exploring their historical and conceptual foundations can lead to more effective instruction. In addition, the study shows how interdisciplinary approaches can enrich student learning by connecting abstract mathematical ideas to practical, everyday situations. These insights support the ongoing improvement of teacher education programs and curriculum design, promoting both specialized knowledge and interdisciplinary competence. While the study points out strengths in the teacher's knowledge and practice, it also identifies areas where growth is still needed, encouraging future research to refine and expand the MTSK model for interdisciplinary teaching contexts.

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Ethics declaration: This study was approved by the Research Ethics Committee at the Rey Juan Carlos University with internal registration number 290220241352024. Prior to the intervention, all participating students were fully informed about the purpose, procedures, and voluntary nature of the study through a participant information sheet, and informed consent was obtained. The participating teacher also provided written consent. To ensure the privacy and confidentiality of all participants, pseudonyms were used, and all data were securely stored and handled. Approval was additionally obtained from the school administration, ensuring compliance with ethical standards related to educational research.

Declaration of interest: The authors declared no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

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