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Research Article

Enacting knowledge in context: A classroom-based analysis of a pre-service teacher's practice

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ABSTRACT

This study explores how a pre-service mathematics teacher mobilizes specialized knowledge while teaching three geometric concepts: similarity, homothety, and Thales' theorem. Drawing on the mathematics teachers' specialized Knowledge model and Duval's (1995) theory of registers of semiotic representation, the study examines how knowledge domains are enacted through multiple representations. Data were collected from three consecutive lessons during the teacher's practicum in a socioeconomically disadvantaged and traditionally structured classroom. Findings indicate that the pre-service teacher evidenced representational fluency and procedural clarity, particularly in the use of diagrams and gestures to convey proportional reasoning. However, conceptual generalizations and formative engagement with students' thinking remained limited. The study underscores the importance of teacher education programs in explicitly linking representational practices with epistemic goals and student reasoning, especially in socioeconomically disadvantaged contexts where systemic constraints often restrict opportunities. This research contributes to ongoing discussions on pre-service teacher development and the pedagogical demands of geometry instruction in authentic classroom settings.

Keywords: specialized knowledge, registers of semiotic representation, teacher education, geometric transformations, procedural fluency

INTRODUCTION

Preparing future mathematics teachers remains a significant challenge, particularly in supporting pre-service teachers to develop both strong content knowledge and the ability to teach for conceptual understanding (e.g., Ball et al., 2008; Hill et al., 2005; Shulman, 1986). In geometry education, this challenge is especially evident, as instruction often emphasizes definitions, formulas, and procedures while giving limited attention to reasoning processes and the coordinated use of diverse representations (e.g., Battista, 2007; Caviedes et al., 2024; Clements, 2003; Freudenthal, 1983). Such practices constrain students' geometric reasoning and spatial thinking, as highlighted in international assessments (OECD, 2023), and expose deeper weaknesses in teacher preparation. Research further shows that many pre-service and in-service teachers share misconceptions with their students, particularly in recognizing geometric properties and understanding relationships between figures (e.g., Avcu, 2022; Seah & Horne, 2020; Sinclair, 2016). This concern has been foregrounded in the 26th ICMI study on advances in geometry education, which underscores the persistent difficulty of fostering meaningful geometric understanding (Lowrie et al., 2024).

Within this landscape, three interconnected topics–homothety, similarity, and Thales' theorem (TT)–offer a valuable lens for investigating teacher knowledge. All draw on proportional reasoning and require the coordination of multiple semiotic registers, including visual, algebraic, and spatial forms (Duval, 2017; Espinoza-Vásquez et al., 2025; Henríquez Rivas et al., 2021; Seago et al., 2014). Although they pose distinct instructional challenges, together they engage core geometric ideas of dilation, transformation, and invariance, which are central to developing reasoning, proof, and representational use in geometry instruction (NCTM, 2000). Examining how teachers mobilize knowledge in these areas sheds light on the broader demands of geometry education, particularly the selection and coherent coordination of representations across diverse teaching contexts (e.g., Mwadzaangati, 2019; Mwadzaangati et al., 2022; Tachie, 2020).

Recent studies have explored task design and instructional strategies for teaching homothety, similarity, and TT (e.g., Aguilera-Moraga et al., 2025; Basu et al., 2022). For example, Basu et al. (2022) show that rich, well-sequenced tasks using nonstandard shapes can deepen students' understanding of similarity, homothety and proportionality. Homothety, a geometric transformation involving proportional scaling from a center, requires the coordination of spatial reasoning and algebraic relationships, demanding fluency across semiotic registers to support conceptual understanding. Although central to transformation geometry curricula, most studies have focused either on student misconceptions (e.g., Oliveira & Lima, 2018; Sarpkaya Aktaş & Ünlü, 2017) or on the use of digital tools (e.g., Araújo & Gitirana, 2022; Gómez-Chacón & Kuzniak, 2015). Belbase et al. (2020) further show that pre-service teachers' beliefs about dynamic geometry software often determine whether learning opportunities become conceptual or remain procedural, while little research has examined how teachers mobilize knowledge in authentic classroom contexts.

Similarity is closely related to homothety but is frequently introduced statically through side ratios and angle congruence, without establishing connections to transformation. Research shows that teachers often default to ratio-based reasoning, rarely engaging students in exploring homothety, invariance, or transformation (e.g., Seago et al., 2013, 2014). Even when representational fluency is present, these deeper conceptual links often remain implicit. TT complements these ideas by providing a structured context for proportional reasoning involving parallel lines and transversals. Beyond its geometric conclusion, TT offers opportunities for integrative reasoning across arithmetic, algebra, and geometry (e.g., Espinoza-Vásquez et al., 2025), yet in practice it is often reduced to empirical verification or visual demonstration, bypassing the deductive reasoning central to geometric proof (Filloy & Lema, 1996).

The knowledge gaps discussed above are often intensified in developing countries, where systemic and material constraints strongly influence classroom instruction (Mwadzaangati, 2019). Increasingly, research emphasizes the importance of studying teaching in real, non-idealized settings, where institutional and socio-economic factors shape practice (Bartell et al., 2017; Ensor, 2001). In such contexts, teachers may display procedural fluency but struggle to connect representations with student thinking or conceptual goals. This is particularly evident in geometry, where fragmented use of representations can hinder engagement with foundational ideas such as transformation and invariance (e.g., Caviedes et al., 2024; Caviedes et al., 2025; Espinoza-Vásquez et al., 2025; Seago et al., 2014). The central challenge, then, is to develop strategic, context-sensitive approaches to support effective geometry teaching. This shift requires moving beyond deficit views to empower teachers to foster exploration using accessible, low-cost tools like paper and pencil (Mwadzaangati, 2024).

Building on these insights, this study characterizes how a pre-service secondary mathematics teacher (PST) mobilized specialized knowledge through semiotic registers across a sequence of geometry lessons. The analysis focuses on how this knowledge is enacted through the coordination of multiple representations and how instructional practices respond to broader epistemic and contextual demands. Rather than evaluating effectiveness, the study examines knowledge-in-use within a resource-constrained classroom. In doing so, it contributes to ongoing discussions in mathematics teacher education on preparing teachers for meaningful and contextually grounded geometry instruction.

THEORETICAL FRAMEWORK

This study draws on two complementary theoretical perspectives to analyze the specialized knowledge mobilized by a pre-service teacher during geometry instruction: the mathematics teachers' specialized knowledge (MTSK) model (Carrillo-Yáñez et al., 2018) and Duval's (1995) theory of registers of semiotic representation.

Mathematics Teachers' Specialized Knowledge

Understanding what teachers need to know to teach mathematics effectively has been the focus of considerable research over the past three decades. Shulman's (1986) concept of pedagogical content knowledge (PCK) and its refinement in the form of mathematical knowledge for teaching (MKT) (Ball et al., 2008) highlight that effective teaching requires a combination of content and pedagogical knowledge. The MTSK model offers a more fine-grained structure that emphasizes the disciplinary specificity of mathematics and the specialized nature of teachers' mathematical work (Carrillo-Yáñez et al., 2018). The model distinguishes between two main domains–mathematical knowledge (MK) and PCK–with further subdomains. This study focuses on three subdomains: knowledge of topics (KoT), knowledge of mathematics teaching (KMT), and knowledge of features of learning mathematics (KFLM).

KoT refers to the teacher's understanding of mathematical concepts, definitions, properties and the relationships that structure the content (Montes et al., 2013). KMT involves the design and selection of explanations, examples, representations, and instructional sequences tailored to students' needs (Carrillo-Yáñez et al., 2013). KFLM involves awareness of how students engage with mathematical ideas cognitively and emotionally, including common misconceptions, difficulties and learning progressions (Carrillo-Yáñez et al., 2018). These three subdomains provide a conceptual structure for identifying the forms of specialized knowledge that become visible in teaching practice.

Registers of Semiotic Representation

Duval's (1995) theory of registers semiotic representation offers a complementary perspective by focusing on how mathematical meaning is constructed and communicated through multiple representations. According to Duval (2006, 2017), two cognitive processes are essential to mathematical understanding: treatment, which refers to operations performed within a single register (e.g., manipulating algebraic symbols), and conversion, which involves transitioning between registers (e.g., symbolic expression into a geometric figure). These processes enable students to access mathematical objects from multiple representations and to build integrated conceptual structures (Caviedes et al., 2024).

Registers can be verbal (natural language [NL]), symbolic (algebraic notation), iconic (figures or sketches), geometric (spatial configurations), or graphical (Cartesian diagrams). In practice, teachers constantly shift between these registers through gestures, spoken explanations, written symbols, and drawn figures to convey mathematical concepts. Duval's (1995) theory provides a framework for analyzing not only what knowledge is activated in teaching but how that knowledge is communicated and made accessible.

Recent research has emphasized the need to address the semiotic and cognitive dimensions of MK, particularly how teachers understand and navigate the challenges students face when interpreting and working with different representations (e.g., Iori, 2018; Presmeg, 2006; Verdugo-Hernández & Caviedes, 2024). For instance, Iori (2018) found that while many secondary teachers show intuitive awareness of the distinction between a mathematical object and its semiotic representations, their understanding often lacks conceptual clarity and reflects non-specialist interpretations of key notions such as representation, treatment, and conversion. This gap constrains their ability to anticipate or respond to students' difficulties in working across multiple semiotic registers and highlights the need to explicitly integrate semiotic reasoning into teacher education programs.

In response to this call, the present study incorporates the semiotic dimension as an essential aspect of what makes mathematics teacher knowledge specialized. Although the study does not formally adopt the networking of theories framework, it draws on its principles (Prediger et al., 2008) to coordinate two complementary perspectives. The MTSK is used to identify and interpret the type of knowledge enacted during instruction, while Duval's (1995) theory serves as a lens for analyzing how this knowledge is constructed, communicated, and transformed through the use of multiple semiotic resources (e.g., Caviedes et al., 2023). For example, when a teacher transitions from a geometric diagram to an algebraic proportion, this conversion not only reflects the coordination of symbolic and geometric representations, but also illustrates the simultaneous activation of KoT, through reference to proportional properties, and KMT, through intentional instructional design that supports student engagement with various forms of representation.

While the combined use of MTSK and Duval's (1995) framework offers robust tools for analyzing instruction, neither fully addresses the sociocultural or institutional factors that shape teaching (Bartell et al., 2017; Ensor, 2001). Nonetheless, their integration provides a focused lens for examining how a PST enacts specialized knowledge in geometry through the coordination of semiotic resources using low-cost materials (e.g., paper and pencil), highlighting the situated nature of teaching in resource-constrained contexts.

METHODOLOGY

General Characteristics of the Research

This study is part of a national project focused on pre-service secondary teachers, particularly in the context of their professional training during the final year of their teacher education program. The participant in this study was enrolled in the tenth semester of a five-year program and completed all academic coursework in mathematics, pedagogy, and didactics. As part of the program's culminating practicum component, the participant undertook 27 hours per week of supervised school-based teaching. Importantly, the program's didactics of mathematics course encompasses key aspects such as the analysis of student difficulties and epistemological obstacles, curriculum organization, and the critical review of educational research. These topics are intended to be linked to major mathematical domains including Number Systems, Algebra, and Functions.

Research Design

This study adopts a qualitative case study methodology to explore how a PST mobilizes specialized knowledge when teaching geometric concepts. Specifically, it follows what Stake (1995) terms an instrumental case study, in which a particular case is examined to gain insight into a broader issue, in this case, the enactment of specialized knowledge in classroom practice. The study is situated within a mathematics education program in a Latin American country (Chile), and the classroom context corresponds to a low-income secondary school characterized by a traditional, teacher-centered instructional model still prevalent in many such settings. The PST was purposefully selected due to the relevance of the lesson content to the study's objectives.

Data Corpus and Unit of Analysis

The data corpus includes video recordings and transcriptions of three consecutive mathematics lessons delivered by the PST during the practicum. A total of eleven lessons were recorded; however, those directly addressing homothety, similarity and TT were selected for analysis. These consisted of three 90-minute sessions covering:

(1) similarity of triangles,

(2) the definition of homothety, including direct and inverse cases, and

(3) a review session incorporating TT.

Non-participant observation was employed (Cohen et al., 2007), allowing researchers to document classroom interactions without influencing the instructional process. Transcriptions include verbal explanations by the PST, student contributions, gestures, drawings, and inscriptions made on the whiteboard. Lesson plans were prepared by the PST and reviewed by the school-based mentor teacher, with the support of a university tutor. While the mentor retained primary responsibility for supervision, the university tutor contributed formative guidance.

The unit of analysis was the PST's instructional activity, delivered to a group of 40 students aged 14–15, and segmented into instructional episodes–defined as coherent sequences of interaction centered on a concept or task (Cohen et al., 2007). Each episode consisted of a series of discursive interventions (by both the PST and the students) sequentially numbered and served as the analytic unit through which both teacher knowledge and semiotic activity were examined.

Table 1. Analytical framework: Subdomains of MTSK and registers of semiotic representation

Constructs/code	Definition	Example
MTSK		
KoT	Accuracy and conceptual depth in the mathematical content	Providing a precise definition of geometric concepts; explaining properties of homothety
KMT	Use of appropriate pedagogical strategies and multiple representations	Selecting illustrative examples; using representations to support explanations
KFLM	Anticipation of student difficulties and connection to prior knowledge	Identifying common misconceptions; adjusting instruction based on observed errors
Registers of semiotic representation		
NL register	Use of verbal explanations and descriptions	Explaining geometric concepts orally using every day or mathematical language
AR	Use of symbols, formulas, and algebraic expressions	Writing proportions or equations to represent relationships between figures
GeR	Use and reference to geometric diagrams and properties	Drawing and analyzing figures such as triangles and parallel lines to illustrate theorems and properties
IR	Use of alternative images, such as visual representations or analogical images	Using simplified sketches or textbook visuals to evoke geometric ideas
GR	Use of gestures and physical movement to illustrate mathematical ideas	Indicating transformations like dilation or direction using hand gestures
Treatment (T)	Transformations within a single semiotic register	Solving an equation or manipulating a figure within (AR) and (GeR) respectively
Conversion (C)	Translation between different semiotic registers	Moving from a geometric figure (GeR) to an algebraic expression (AR) to represent similarity

Analysis

Data were analyzed using a deductive-inductive approach, structured around a cyclical process of segmentation, coding, comparison, and refinement. Full transcripts of the three selected lessons were systematically segmented in Excel spreadsheet into instructional episodes. The qualitative analysis was conducted in two stages, integrating principles of thematic and interpretive analysis suited to classroom-based inquiry (Burns & Grove, 2004). In the first stage, instructional episodes in which specialized knowledge became visible were identified and analyzed using the MTSK model (KoT, KMT, and KFLM) (see **Table 1**). In the second stage, the same episodes were examined through Duval's (1995) theory of registers of semiotic representation with attention to the use of NL, algebraic, geometric, iconic, and gestural registers, as well as the cognitive operations of treatment and conversion between registers (see **Table 1**).

A manual coding scheme was developed to operationalize constructs from both frameworks, including explicit definitions, guiding indicators, and illustrative examples to support consistency and analytic clarity. To enhance reliability, all episodes were independently coded by both researchers. The process was cyclical and iterative: initial coding was followed by collaborative review, refinement of categories, and re coding when necessary. This procedure supported progressively aligned interpretations and enabled a robust analysis of how specialized teacher knowledge was mobilized and communicated through the coordination of representations in authentic classroom practice.

First session: Triangle similarity

The first session focused on introducing triangle similarity through proportional reasoning and similarity criteria. The PST began with a brief review of proportionality and then stated the lesson objective: "Identify criteria for triangle similarity and apply them to given pairs of triangles". To activate students' prior knowledge, the PST asks, "*Do you know what similarity is?*" initiating a dialogue with the students. The PST addressed concepts such as congruence and similarity in triangles, defining them as: *"Two figures are congruent when they have the same shape, the same dimensions, and the same angles … and … two figures are similar when they have the same shape, the same angles, but their measurements are not equal."* This phrasing gives indications of KoT, specifically an operational understanding of similarity and congruence grounded in visual attributes. However, the framing remains static and visual, without any mention of dynamic or transformational approaches. The lack of reference to transformation may reflect limitations in conceptual depth.

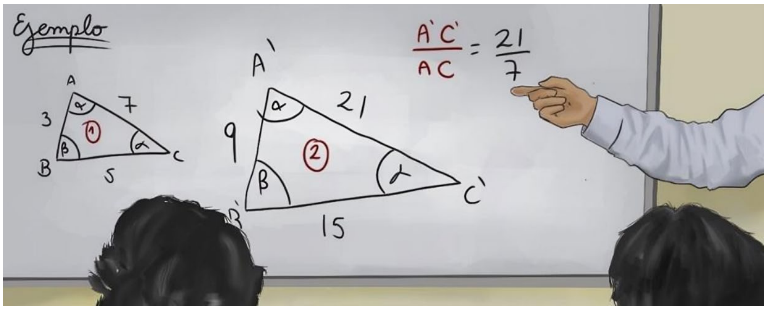

Figure 1. Activity proposed by the PT (the authors' compilation)

The PST then linked similarity to proportion, describing a ratio as: "*a comparison between two or more quantities*," and connects this idea with proportion, defined as "*an equality between two or more quantities*." These definitions were mathematically imprecise–particularly in describing a proportion as an "equality between quantities" rather than between ratios, which may lead to conceptual ambiguity. Nevertheless, the phrasing suggests an attempt to connect students' prior arithmetic understanding with the new geometric content. This illustrates a common challenge in mathematics instruction: navigating the shift from informal, everyday language to formal mathematical discourse. The episode gives indications of limitations in the PST's KoT, particularly in articulating definitions with mathematical precision. Despite this, the PST used these definitions as a bridge into a practical example involving triangle side lengths, coordinating the geometric register (GeR) through board drawings, NL for explanation, and the algebraic register (AR) to construct and manipulate ratios. Subsequently, the PST drew two triangles on the board and introduced a worked example involving specific side lengths:

> 1 PT: "Now we're going to compare … just like a ratio. Let's say, for example, 21 is to 7. If I have here A'C, which side am I going to compare it with?"
>
> 2 S5: "With A'C'."
>
> 3 PT: "With A'C' […] Here we establish a ratio; we are comparing two quantities, two measurements, okay? We're comparing one side of a triangle with its corresponding side, that is, the side that, when looking at the figures, corresponds to it because it is formed by the same angles, okay? [Writes ratio A'C'/AC] […] So, if we compare the sides to obtain the ratio, how do we calculate the ratio? What is the value of a ratio?"
>
> 4 S3: "Its quotient."
>
> 5 PT: "Very good, so 21 divided by seven, how much is it?"
>
> 6 S3: "Three."
>
> 7 PT: "That's the ratio, the ratio we obtained by comparing [writes] […] So now, we've established our first ratio; we find that these triangles are at a ratio of three."

This segment involved conversions between the GeR, AR, and NL. The use of side labeling and proportion writing on the board (**Figure 1**) supported procedural identification of corresponding sides and their numerical relationships. The coordination of registers gives indications of KoT and KMT, as the PST structured a visual-symbolic path to quantify similarity. However, while the register use was appropriate, the mathematical focus remained procedural. The concept of why this ratio validates similarity was not discussed. For example, student 3's response ("three") was accepted without further discussion. This absence indicates minimal enactment of KFLM, as the PST did not address possible misconceptions or alternative explanations.

Later, the PST stated: "*you must compare the longest side with the longest side, the shortest side with the shortest side*" This instruction was accompanied by written algebraic proportions (AR) and gesture-based reference to the labeled triangles (IR-GeR) (**Figure 1**). The PST's explanation gives further indications of KoT, particularly in referencing invariant properties of similar figures such as shape and angle preservation. The

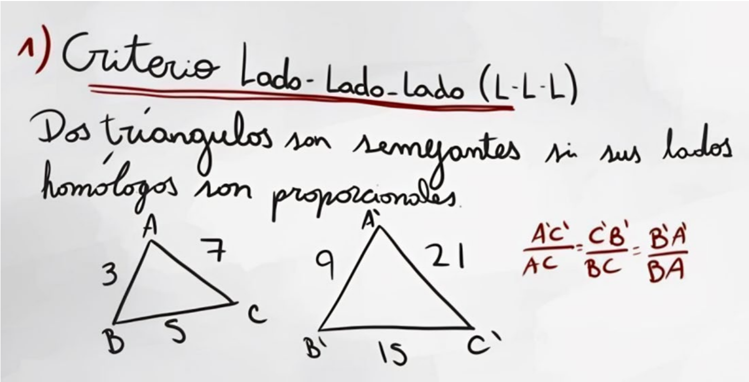

Figure 2. SSS criterion (the authors' compilation)

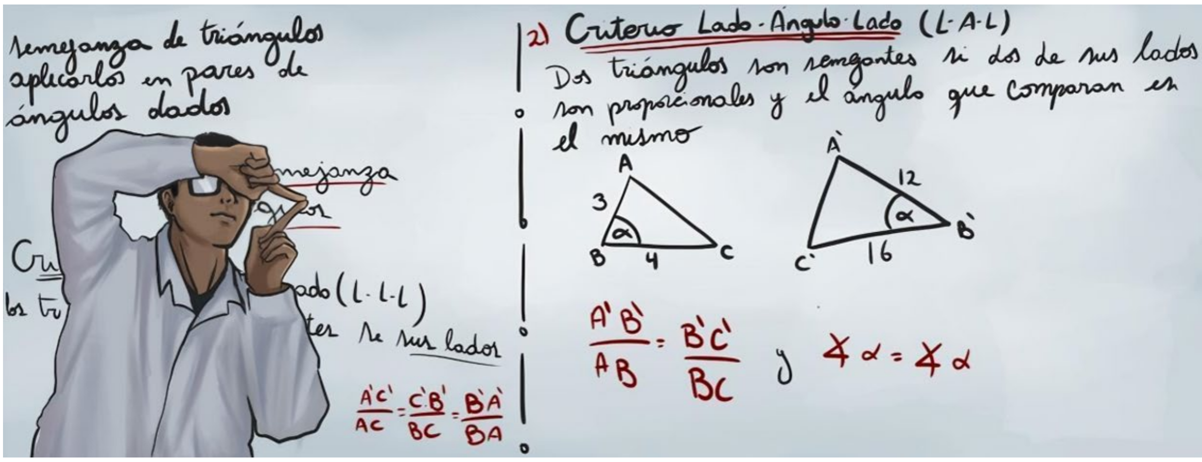

Figure 3. SAS criterion (the authors' compilation)

repeated matching of side types suggests attention to visual proportionality, although the concept remained undeveloped in theoretical terms. The coordination of registers might provide an indicator of the PST's KMT, as it supported clarity in identifying corresponding parts. However, this segment focused on reinforcing rule-following over concept exploration.

Following this introduction, the PST transitioned into the similarity criteria. The first was the side-side-side (SSS) criterion: "*If the three sides of one triangle are proportional to the three sides of another, the triangles are similar*." This explanation was accompanied by a diagram on the board illustrating two triangles with proportional side markings (GeR), written ratios comparing the side lengths (AR), and verbal elaboration (NL) (**Figure 2**). The coordination across these registers gives indications of the PST's KoT, particularly in applying the proportionality condition correctly. The structured and sequential use of representations may evidence KMT, as it facilitates clarity and alignment between the visual and symbolic components. However, the explanation was declarative, and no exploration of the underlying reasoning behind the criterion was offered. Students were not prompted to generate examples, question the sufficiency of the condition, or compare it, which limited conceptual engagement.

The second criterion, side-angle-side (SAS), was introduced as: "*Two triangles are similar if two of their sides are proportional and the included angle is the same".* To support this explanation, the PST incorporated gestures (IR) and a triangle sketch (GeR) in which the right angle between two sides was emphasized. The PST used a hand gesture–bringing index fingers together–to illustrate the position of the included angle (**Figure 3**). This representational move gives indications of KMT, as it attempted to visually highlight the spatial configuration relevant to the criterion. The integration of gestures with symbolic and visual elements suggests attention to multimodal clarity. However, the PST did not engage students in verifying whether the position of the angle matters, nor did the explanation address potential misconceptions about angle placement, which are co*mmon* in similar tasks. As such, while the explanation was technically correct, it lacked opportunities for elaboration or formative assessment.

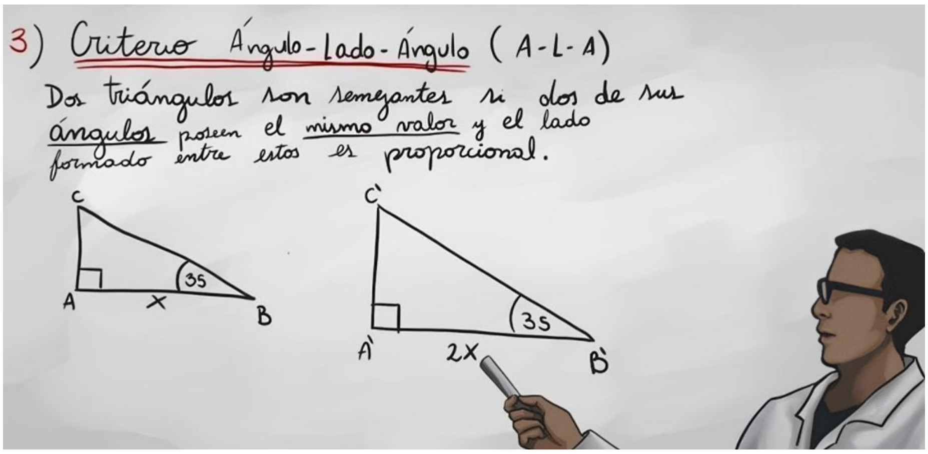

Figure 4. ASA criterion (the authors' compilation)

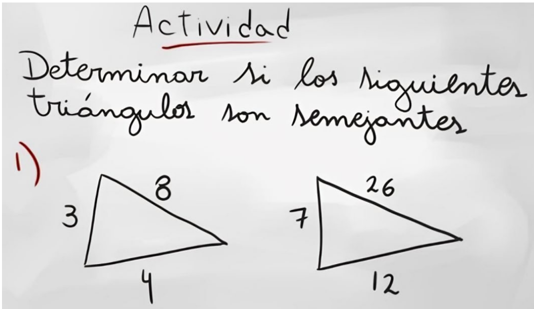

Figure 5. Activity provided by the PT to the students (the authors' compilation)

The third and final criterion presented was angle-side-angle (ASA). The PST used a labeled triangle on the board, marking two equal angles and a proportional included side. One side was labeled with a variable (x) to indicate proportional scaling. The PST explained: "I*f two angles are equal and the included side is proportional… this side is double the other one, so the ratio is 2*." This explanation involved GeR, AR, and NL simultaneously (**Figure 4**). The coordination of symbolic notation and diagrammatic representation gives further indications of KoT, particularly in recognizing that similarity can be justified through angle congruence and proportionality. The use of the variable and ratio evidenced the PST representational fluency. This episode might indicate PST's KMT, as it showed intentional alignment between algebraic scaling and geometric structure. However, similar to the previous cases, the presentation did not include comparisons across criteria. The PST did not ask whether two angles alone were sufficient, nor did he contrast ASA with other cases, which limited opportunities for deeper conceptual analysis, thus giving little indication of KFLM.

Finally, the PST presented a short task in which students were asked to determine whether two given triangles were similar (**Figure 5**). The task required students to identify the appropriate similarity criterion based on the information provided. The PST facilitated the activity through brief verbal prompts:

> 8 PT: "If two triangles are similar, you can't just use any of the three criteria. No. You need to look at the information carefully. Are they similar?"
>
> 9 S1: "No."
>
> 10 PT: "Side-side-side."

This exchange gives indications of KMT, as the PST attempted to direct students' attention to the sufficiency of the given data and the correct application of the similarity criteria. While the question was closed and did not elicit elaborate reasoning, it suggests some awareness of the need to distinguish between the criteria based on available information. Furthermore, by foregrounding that "you can't just use any of the three," the PST implicitly acknowledged that students might overgeneralize or misapply similarity conditions–a point that gives indications of KFLM. However, the instructional move stopped short of exploring this issue in depth. The

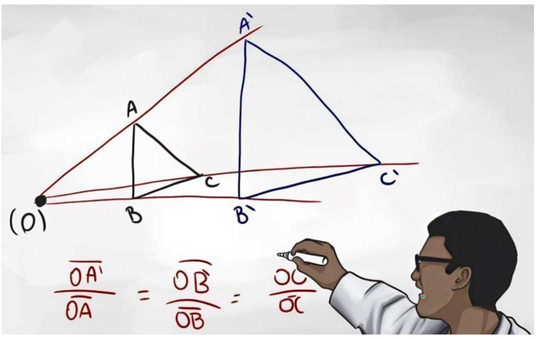

Figure 6. Example of the definition of homothety (the authors' compilation)

student's one-word response was accepted without follow-up, and the opportunity to engage in justification or discussion was not pursued. As such, while there is evidence of representational and procedural fluency, the interaction remained directive (**Figure 5**).

Second session: Definition of homothety and direct and inverse homothety

The second observed session focused on introducing and exploring the concept of homothety through six cases illustrating different values of the homothety ratio (K). The PST began by stating the lesson's objective: *"Understand the concept of homothety and its associated elements."* The session followed a structured progression, with emphasis on visual and algebraic representations to convey the transformation properties of homothetic figures. The PST defined homothety as a "*geometric transformation by which a geometric figure changes its dimensions according to a scale factor, which we call the homothety ratio*." This verbal definition was accompanied by a freehand sketch of two similar triangles connected to a shared center of homothety, labeled O (**Figure 6**). The use of NL in tandem with geometric drawings (GeR) gives indications of KoT, specifically in recognizing that homothety preserves shape while altering size. However, the explanation remained descriptive rather than conceptual, with no reference to underlying transformational principles such as dilation, which limits the theoretical depth.

To anchor the definition, the PST introduced a numerical example: "$\underline{AB}$ = *3 cm*, $\underline{BC}$ = *3 cm*, $\underline{CA}$ = *3 cm*, and $\underline{A'B'}$ = *6 cm*, $\underline{B'C'}$ = *6 cm*, $\underline{C'A'}$ = *6 cm*." To illustrate the homothetic transformation, the PST explained that by applying the scale factor K to these measures $\underline{OA}$ = *2.3 cm*, $\underline{OB}$ = *4.5 cm*, and $\underline{OC}$ = *2.6 cm*, the new segment lengths $\underline{OA}$, $\underline{OB}$, and $\underline{OC}$ were obtained. This numeric scaling was supported by visual markings on the triangle sides and proportional comparisons, coordinating GeR with the AR. The explanation gives further indications of KoT and KMT as it employs familiar values to illustrate the effect of scaling. Next, the PST explained how to determine the homothety ratio by calculating distances from the center of homothety to each vertex, stating: *"To calculate the scale factor, we divide the distance between the center of homothety 'O' and a homothetic point by the distance between 'O' and the corresponding original point."* This explanation coordinated AR and GeR with NL, and gives indications of both KoT and KMT, as it transformed an abstract idea into a practical procedure through multiple registers. The PST then initiated a guided dialogue to introduce the concepts of direct and inverse homothety. The interaction provided a moment of verbal engagement, where students offered tentative definitions that were then refined by the PST. However, despite the presence of student responses, the transcripts reveal instances where the PST did not acknowledge or build upon student contributions. In some of the exchanges (17–19), the PST overlooked some responses, missing opportunities to validate or clarify student thinking. These omissions reflect limitations in KFLM, particularly in terms of dialogic engagement and formative assessment.

> 11 PT: "What is a homothety?"
>
> 12 S1: "A numerical transformation where the figure changes."
>
> 13 PT: "Right, as S1 says, it's a geometric transformation where a figure changes. S2, what can happen to the figure? Does it enlarge or shrink?"
>
> 14 S2: "It enlarges or shrinks."

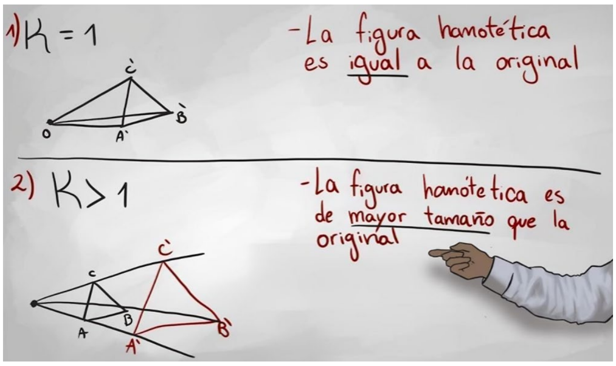

Figure 7. First and second example of the homothety ratio (the authors' compilation)

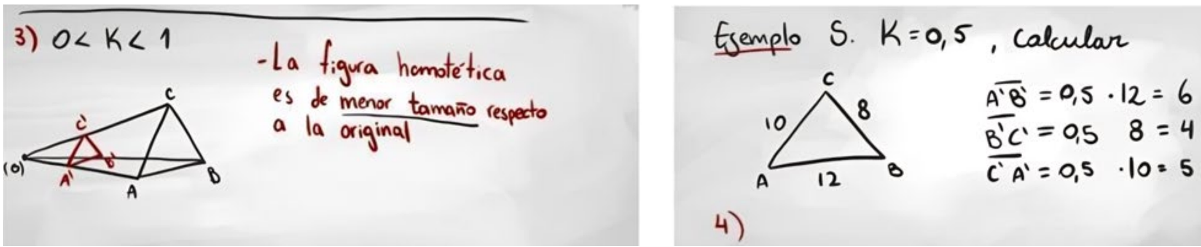

Figure 8. Third example of the homothety ratio (the authors' compilation)

> 15 PT: "Exactly, it increases or decreases in size […] And what do we call that?"
>
> 16 S3: "Proportional."
>
> 17 PT: "Proportional, correct? Very good. So today, we will look at two types of homothety: direct and inverse. What determines if it is direct or inverse? It depends on a factor we studied. What is the factor that makes the figure increase or decrease in size?"
>
> 18 S4: "Homothety."
>
> 19 PT: "The scale factor, which is the value we multiply by. What range of numbers can this factor take?"

This exchange gives indications of KMT, as the PST used sequential questioning to guide students toward identifying the scale factor (K) as the determinant of homothetic transformation. It also suggests some attention to KFLM, as the PST responded flexibly to students' informal or imprecise responses and reoriented them toward the desired mathematical vocabulary and concept. The coordination of NL, AR, and GeR within this dialogic context highlights the semiotic dimension of knowledge mediation.

The remainder of the session was organized around six cases, each exploring a specific range or value of K. The PST consistently employed diagrams, labels, and numerical values to illustrate each scenario, with register coordination across AR, GeR, and occasionally gestural representations (IR).

Case 1 and 2: K = 1 and K > 1: The PST stated that "*The homothetic figure is identical to the original*" when K = 1, and "*larger than the original*" when K > 1. Diagrams (GeR) showed side lengths increasing proportionally (**Figure 7**). This segment gives indications of KoT, and the coordinated use of visual and verbal registers provide indications of KMT. The explanation supported procedural clarity but did not elaborate on underlying geometric properties or invite student interpretation.

Case 3: 0 < K < 1: In this case, the PST drew a reduced triangle in red and wrote "T*he homothetic figure is smaller than the original*" (**Figure 8**). The visual use of color and relative size supported meaning-making and might provide an indicator of KMT, particularly in using visual strategies to convey proportional reduction. The explanation also gave indications of KoT by connecting fractional scale factors with dimensional reduction. However, as in previous cases, the discussion centered on descriptive outcomes–what happens–rather than explanatory reasoning about why the transformation behaves this way.

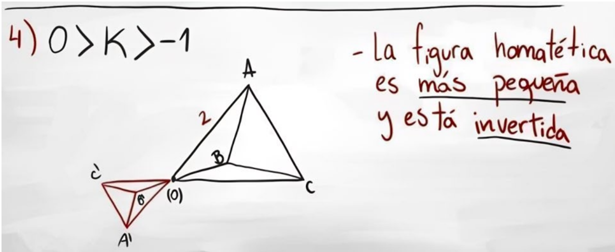

Figure 9. Fourth example of the homothety ratio (the authors' compilation)

Case 4: 0 > K > –1: To introduce the concept of a negative homothety ratio, the PST assigned a value of 2 units to the OA segment and applied a homothety ratio of K = –0.5. The PST physically showed the transformation using IR–mimicking a shortening of the segment–and drew the inverted image (**Figure 9**). The PST stated: "*The negative sign indicates that the figure flips.*"

This explanation involved conversion between AR, GeR, and IR, giving indications of both KoT and KMT. Additionally, the explanation anticipated the common misconception of "negative distance," suggesting emerging KFLM. The following exchange occurred during this explanation:

> 20 PT: "The distance between O and A is two units, and let's suppose K is –0.5. What is 2 multiplied by –0.5."
>
> 21 S4: "It's –1."
>
> 22 PT: "–1, right? Can we have negative distances?"
>
> 23 S5: "No."
>
> 24 S4: "No, it flips."
>
> 25 PT: "Exactly. We project this line segment [points to the two-unit segment], and the new length is our result. If it was 2, now it's 1. Distances can't be negative. The negative sign indicates that the figure flips. So, point A' is here [draws on the board]."
>
> 26 PT: "So, if we know that for negative values the figure flips, what happens for values less than –1."
>
> 27 S10: "The same, but larger."
>
> 28 PT: "Exactly. The figure will be larger and inverted. But since it's negative, it tells us that the orientation will flip. Therefore, for values less than –1 the figure will be larger and inverted [draws on the board]."

This sequence gives indications of KoT, particularly in identifying the dual impact of negative homothety ratios–both inversion and scaling. The use of coordinated AR (–0.5), GeR (inverted triangle), IR (hand gestures to show shortening), and NL supports conversion between symbolic, visual, and embodied representations. The explanation gives indication of KMT, as the PST reinforced the idea that the sign of K determines orientation, while its magnitude affects size. Additionally, the dialogue reflects emerging KFLM, as the PST anticipated and addressed a typical student misconception–whether negative distances are meaningful in geometry. The strategy of physically exemplifying and then redrawing the transformation provided a concrete, multimodal anchor for interpreting negative homotheties.

Case 5: K < –1: This case extended the exploration of negative homothety ratios by focusing on the combined effects of sign and magnitude. The PST guided the discussion using both questions and board-based visual representations to clarify how values inferior to –1 invert and enlarge the figure.

> 29 PT: "So, for values where K is between 0 and –1 … the homothetic figure is smaller and inverted. Now, what happens for values less than –1?"

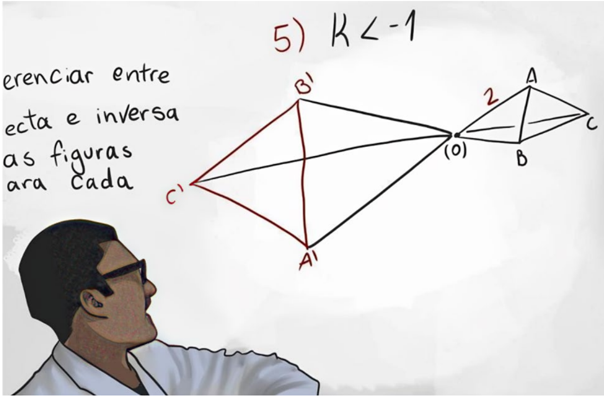

Figure 10. Fifth example of the homothety ratio (the authors' compilation)

> 30 S10: "The same, but larger."
>
> 31 S4: "The figure shrinks more."
>
> 32 PT: "Does it shrink?"
>
> 33 S10: "When it is a decimal number less than –1, it gets larger."
>
> 34 PT: "[…] if I have a value multiplied by something negative, the greater the absolute value of the negative number, the farther it will be from zero. Suppose the center of homothety is my zero. If the distance from 0 to A is 2 units, and I apply a homothety ratio, for example, of –100, it means that from 0 to A there are –100 units. But we said that homothety involves straight lines, so this line cannot have –100 in that direction [pointing towards A] because that direction only accommodates positive values. Therefore, it must be represented from 0 in the opposite direction. That's why it inverts."

This interaction gives indications of KoT, particularly in the PST's attempt to unpack the mathematical relationship between the homothety ratio's absolute value and its geometric effects. By posing questions about relative size and prompting clarification, the PST also engaged in pedagogical mediation indicating his KMT. The use of a hypothetical extreme value (K = –100) and the visual pointing toward the direction of projection suggested an effort to anchor abstract symbolic reasoning (AR) in concrete spatial understanding (GeR). The PST further integrated NL and gestures (IR) to describe directionality and orientation reversal, creating a multimodal learning experience.

Moreover, the PST also gave indications of KFLM, as he responded to contradictory or inaccurate student claims (e.g., "the figure shrinks more") and redirected them toward a correct interpretation. **Figure 10** depicted the result as larger and inverted, reinforcing the idea that transformations under K < –1 result in a reversal of orientation and an increase in size. This explanation aligned algebraic, verbal, and visual elements, offering multiple points of access to the underlying geometric concept. The emphasis on directionality, inversion, and proportional scaling highlighted how semiotic conversions across AR ↔ GeR ↔ IR were used to support the explanation.

Case 6: K = –1: The final case introduced the special instance of a homothety ratio equal to −1. This transformation preserves size but inverts orientation. The PST concluded the typology with a short dialogue, reinforcing the pattern established in earlier examples:

> 35 "PT: So, for values where K < –1 the figure becomes larger and inverted … what happens for K = –1?"
>
> 36 S4: "It's the same, but inverted."

This interaction gives indications of KoT, as it addressed the particular effect of a homothety ratio with magnitude –1. The PST coordinated AR and GeR by drawing two triangles of equal size but with reversed

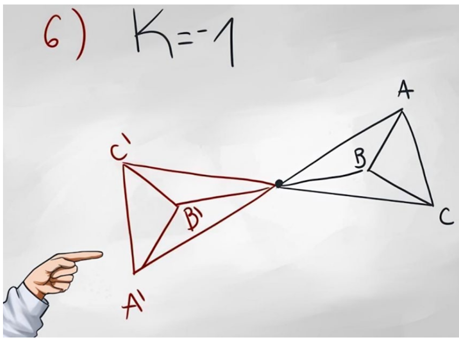

Figure 11. Sixth example of the homothety ratio (the authors' compilation)

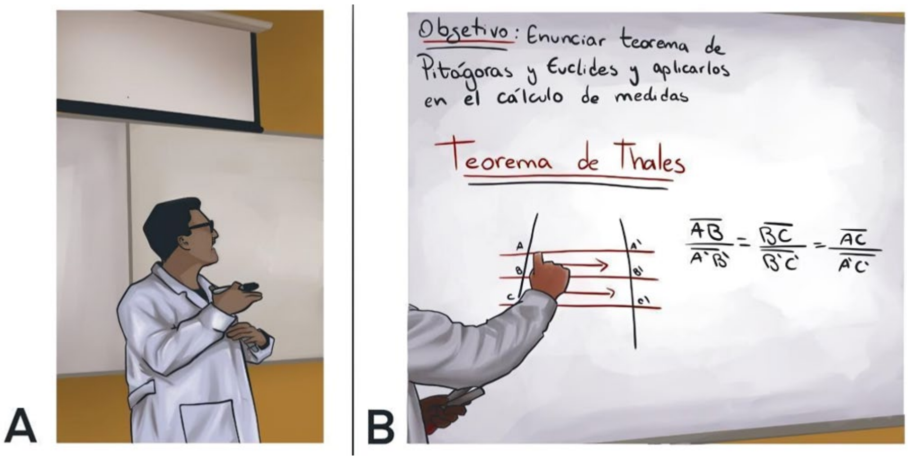

Figure 12. Representation of TT (the authors' compilation)

orientation on the board (**Figure 11**). This conversion between registers offered a visual anchor to support student interpretation. The prompt encouraged brief student engagement, and the PST confirmed the response without further elaboration. While the explanation was procedurally correct, no generalization was made regarding the homothety ratio's behavior across categories, nor was a link drawn to transformational geometry more broadly. The interaction gives indications of KMT, particularly in reinforcing the classification logic of the homothety types. However, no further conceptual questioning or reflection was introduced. Although the use of NL, GeR, and AR supported semiotic coordination, the activity remained largely illustrative. The PST did not use this closing case to synthesize patterns or invite student conjecture, which limited the opportunity to further explore KFLM.

Third session: Feedback on Thales' theorem

The third session focused on introducing and reviewing TT, as well as drawing initial connections to Pythagorean and Euclidean theorems, although the latter were not analyzed in detail. The PST began by declaring the lesson's objective: "*State Thales' theorem and apply it to the calculation of means and introduce Pythagoras' and Euclid's theorems.*" This session extended the topic of similarity and proportional reasoning by emphasizing TT as a foundation for understanding geometric relationships between parallel lines and transversals.

The PST first states TT as "a *proportional relationship between segments formed by two or more parallel lines intersected by two or more transversals.*" While the phrasing was accessible, it lacked formal mathematical precision and did not refer to triangle similarity, which underpins the theorem's justification. This definition gives indications of KoT, particularly in recognizing the proportional relationship among segments, but the absence of a conceptual derivation from similarity (as in prior sessions) suggests limitations in theoretical integration.

To illustrate the theorem, the PST drew a diagram showing three horizontal lines intersected by two transversals, labeling them "$L_1 \parallel L_2 \parallel L_3$ *as parallel lines, and* R_1, R_2 *as transversals*" (part A in **Figure 12**).

As the PST explained the parallelism, he used hand gestures (IR) to mimic the equal spacing of the lines, reinforcing the structural relationship. The coordination between gestures (IR) and the GeR gives further indications of KMT, as these representations helped convey the spatial structure of the situation. This was followed by a visual explanation of segment ratios using symbolic expressions. The PST wrote: "AB / A'B' = BC / B'C' = AC/A'C'" and connected the algebraic form with the geometric configuration. This segment involved a conversion between the AR and GeR, suggesting coordination of representations to support interpretation. The explanation gives further indications of KoT and KMT, respectively–in constructing proportional relationships among corresponding segments and in referencing those segments through labeled diagrams. The PST used labeled side lengths to guide students in identifying and comparing corresponding parts, supporting both conceptual understanding and instructional clarity. The following actions suggest a treatment transformation within GeR, as the PST drew two similar triangles formed by the intersections, emphasizing their shared angles and proportional sides. The use of visual highlighting and side labeling suggested an effort to link TT back to similarity, although the connection remained implicit. This missed opportunity to explicitly reference the similarity criteria from Session 1 limited the conceptual continuity of the teaching sequence.

To guide student interpretation, the PST returned to gestures and spatial explanations, emphasizing parallelism using arm movements and then tracing corresponding segments on the board. This multimodal approach involved coordinated use of IR, NL, and GeR, which may indicate KMT, as the PST reinforced attention to invariant relationships through semiotic integration. However, no direct questions were posed to elicit student reasoning, nor were students asked to validate the proportionality claims or provide justifications. As a result, KFLM was minimally evident in this episode.

Later in the session, the PST generalized: *"Thales' theorem tells us that corresponding sides are proportional."* This phrasing gave indication of KoT, as it summarized the procedural implication of the theorem but again omitted a geometric justification. The use of algebraic relationships to assert proportionality, without reference to the geometric transformations or angle relationships involved, reinforces the session's procedural emphasis. The PST coordinated registers effectively and used multiple representations to support interpretation. However, similar to prior sessions, the lesson focused primarily on presenting and illustrating the content, without engaging students in exploratory reasoning, justification, or comparison.

RESULTS AND DISCUSSION

In the first session, the PST introduced triangle similarity by defining congruence and similarity through NL and supported the definitions with algebraic (AR) and geometric (GeR) representations. While this initial exposition evidenced procedural clarity, the definitions contained inaccuracies–e.g., describing a proportion as an "equality of quantities"–which suggest limitations in KoT. Such phrasing may reflect instructional tendencies where informal or everyday language is relied upon in lieu of more rigorous definitions–possibly due to limitations in content-specific pedagogical training, as observed in other contexts (Seago et al., 2013). The lesson continued with a worked example involving the ratio of corresponding triangle sides. The PST coordinated GeR, AR, and NL to set up and solve proportions between labeled side pairs and emphasized the idea of "matching the longest side with the longest side" (**Figure 1**). This coordination gives indications of both KoT and KMT, as the PST supported procedural fluency with visual reinforcement, but did not engage students in reasoning about why such proportional relationships hold. There was little dialogic exchange beyond checking answers, limiting KFLM enactment. Subsequently, the PST introduced the three standard similarity criteria–SSS, SAS, and ASA–through diagrams (GeR), algebraic ratios (AR), gestures (RFI), and verbal elaboration (NL). Each criterion was explained procedurally and illustrated through labeled triangle sketches. The explanations were technically correct but remained declarative and teacher centered. Students were not invited to compare cases, justify criteria, or explore conceptual underpinnings such as dilation or invariance. This lack of conceptual elaboration indicates a limited mobilization of KFLM. The lesson concluded with a short task asking students to determine whether given triangles were similar. The PST guided the task through closed questions, emphasizing the selection of the appropriate criterion. While the PST correctly anticipated common misconceptions (e.g., applying criteria arbitrarily), there was minimal follow-up on student reasoning or alternative interpretations. **Table 2** summarizes the findings of this session.

Table 2. PST's mobilized knowledge in session 1

Episode/segment	MTSK subdomains	Registers used	Semiotic activity	Analytical notes
Definition of similarity and congruence	KoT	NL	Treatment (within NL)	Static framing; lacks transformational aspect
Proportion as "equality of quantities"	KoT	NL, AR	Treatment (within NL) Conversion (NL → AR)	Imprecise mathematical language; reflects procedural orientation
Worked example with side lengths	KoT, KMT	GeR, AR, NL, GR	Conversion (GeR ↔ AR ↔ NL)	Clear visual-symbolic alignment; lacks conceptual justification
Instruction on comparing sides	KoT, KMT	AR, GeR, NL	Treatment (within NL) Conversion (GeR ↔ AR)	Procedural matching emphasized; no exploration of underlying propositions
SSS, SAS, and ASA criteria explanations	KoT, KMT	GeR, AR, NL, RFI	Conversion (AR ↔ GeR)	Multimodal clarity; limited student interaction or comparative reasoning
Application task: Identifying correct criterion	KMT, KFLM	GeR, NL	Conversion (GeR ↔ NL)	Anticipates overgeneralization; minimal probing of student thinking

Table 3. PST's mobilized knowledge in session 2

Episode/segment	MTSK subdomains	Registers used	Semiotic activity	Analytical notes
Definition and diagram	KoT	NL, GeR	Treatment (NL), Treatment (GeR)	Conceptual basis present; no dynamic framing
Example with numeric ratios	KoT, KMT	AR, GeR	Conversion (AR ↔ GeR)	Ratio interpretation supports procedural fluency
Direct vs. inverse homothety	KoT, KMT, KFLM	NL, AR, RFI	Conversion (NL ↔ AR), RFI use	Some elicitation; student thinking not deeply engaged
Cases for varying K values	KoT, KMT	AR, GeR, RFI	Conversion (AR ↔ GeR ↔ RFI)	Effective visualization; conceptual links not fully developed

The second session focused on the concept of homothety and the interpretation of the homothety ratio (K). The PST introduced homothety as a transformation that modifies a figure's dimensions according to a scale factor (K). The lesson began with a verbal definition supported by a hand-drawn diagram (GeR) featuring triangle ABC and its image A'B'C', with center O. This illustration enabled students to visualize geometric relations, and the alignment between NL and GeR provided indications of KoT. To operationalize the definition, the PST assigned numerical values to segments (e.g., AB = 3 cm, A'B' = 6 cm) and computed the ratio K = 2. This example linked algebraic (AR) and GeR and showed the PST's representational fluency. The PST also explained how the direction and magnitude of K determine whether a transformation is direct (same orientation) or inverse (inverted), using gestures (RFI) and guided questioning to involve students in identifying patterns, giving indications of both KMT and initial KFLM.

To build procedural fluency, the PST presented five cases of homothety based on different values of K. Each case involved conversions between algebraic (AR), geometric (GeR), and iconic (IR) representations, emphasizing proportional measurement and directionality. These semiotic transitions supported visual clarity and rule-based application (KoT and KMT), particularly in distinguishing between direct and inverse transformations. While the approach facilitated procedural accuracy, opportunities for conceptual exploration, such as linking homothety to similarity or discussing geometric properties in coordinate systems, were not developed (Seago et al., 2013). The instructional approach remained largely expository, and although the PST anticipated difficulties with negative values of K, there was limited evidence of formative assessment or dialogic feedback (KFLM). Nonetheless, the PST did offer moments of instructional interaction aligned with KFLM. For instance, when a student incorrectly claimed that "the figure shrinks more" under K < –1, the PST redirected this reasoning by emphasizing that such transformations result in figures that are both enlarged and inverted. This explanation–supported by **Figure 10**–coherently aligned algebraic, verbal, and visual elements, offering multiple entry points to the geometric concept. The emphasis on directionality, inversion, and proportional scaling showcased the purposeful conversion across AR ↔ GeR ↔ IR registers to reinforce the explanation. **Table 3** summarizes the findings for this session.

The final session addressed TT and its applications. The PST defined TT using NL as "a proportional relationship between segments formed by two or more parallel lines intersected by two or more transversals." The verbal description was paired with a hand-drawn diagram of parallel lines and transversals labeled with

Table 4. PST's mobilized knowledge in session 3

Episode/segment	MTSK subdomains	Registers used	Semiotic activity	Analytical notes
Definition of TT	KoT	NL	Treatment (NL)	Descriptive definition; lacks deductive grounding or reference to parallelism
Diagram and gesture-based explanation	KoT, KMT	GeR, RFI, AR	Conversion (GeR ↔ AR); RFI support	Visual-symbolic alignment; reinforces procedural calculation
Segment proportion equalities	KoT, KMT	AR, GeR	Conversion (AR ↔ GeR)	Algebraic formalization; does not support conceptual justification
Application task	KMT, KFLM	GeR, NL	Treatment (GeR); Conversion (GeR ↔ NL)	Procedural accuracy; lacks dialogic or exploratory engagement

segment lengths (GeR). This explanation was descriptive but did not mention the conditions necessary for TT–particularly parallelism or its deductive basis. Thus, KoT was present, but conceptual depth was limited. Using gestures (RFI), the PST emphasized the idea of "equal separation" between lines and illustrated how corresponding segments form ratios. These were subsequently written as algebraic equalities (e.g., AB/A'B' = BC/B'C')–a conversion from GeR to AR.

The PST moved between these registers fluidly but focused on the procedural use of TT to compute unknown segment lengths. While the PST effectively coordinated diagrams and algebraic notation, the lesson lacked dialogic engagement. Students were not prompted to justify why TT holds or to consider the geometric conditions–such as parallelism–under which it applies. Furthermore, the empirical focus on segment measurement took precedence over theoretical reasoning, reflecting patterns previously identified in the literature (e.g., Espinoza-Vásquez et al., 2025). The PST did not draw connections between TT and previously introduced topics (e.g., similarity), missing an opportunity to foster mathematical coherence and connections (e.g., Rodríguez-Nieto et al., 2023). Although the representations were accurate, they served to reinforce rule-based computation. The lesson included one student task involving labeled lines, where learners were asked to compute missing values using TT. The PST confirmed answers but did not provide alternative strategies or conceptual explanations, offering limited evidence of KFLM. **Table 4** shows the PST's mobilized knowledge in session 3.

Across the three sessions, the PST mobilized KoT and KMT through effective coordination of semiotic registers. Representational transitions–particularly GeR ↔ AR and GeR ↔ NL–were central to how mathematical ideas were structured and transmitted (in line with Verdugo-Hernández & Caviedes, 2024). However, the enactment of KFLM was sporadic. While occasional gestures and questions suggested awareness of student challenges, there was little follow-up probing or adaptation of instruction. Instruction across sessions was expository, visually clear, and procedurally sound, but largely teacher centered. Student contributions were rarely expanded or challenged. The lack of explicit connections across sessions–e.g., from similarity to homothety to TT–further limited the development of structural understanding. This pattern reflects structural characteristics of traditional educational contexts, particularly in under-resourced settings where instructional innovation may be limited (Bartell et al., 2017; Ensor, 2001).

Table 5 offers a structured overview of how the PST enacted subdomains of MTSK across the three observed lessons but also underlying properties and connections that were not emphasized. Notably, while KoT and KMT were mobilized, the articulation of KFLM was more limited and often implicit, aligning with prior findings on the challenges PSTs face in anticipating and addressing student thinking (Espinoza-Vásquez et al., 2025; Seago et al., 2014; Verdugo-Hernández & Caviedes, 2024). **Table 5** also highlights how semiotic work–particularly the use of algebraic (RA), geometric (RGe), numerical (RN), and iconic (IR) registers–served as key mediators of the PST's instructional strategies and conceptual framing across all three mathematical topics.

Table 5. PST' specialized knowledge across sessions

Session	Subdomain	Indicator
1	KoT	*Definitions:* Similarity of triangles and types of similarity. *Justifications:* Use of proportions to justify triangle similarity. *Properties:* Invariance of shape and angles in similar figures. *Representations:* Use of AR (algebraic), NR (numerical), GeR (geometric), and IR (iconic) registers to illustrate and communicate similarity. *Procedures:* Calculation of the similarity ratio between corresponding sides. *Connections:* Conceptual links between similarity and proportion.
	KMT	*Teaching strategies:* Integration of IR and GeR to support verbal explanations and visually communicate key relationships. *Guided questions:* Activation of prior knowledge, clarification of misconceptions, and structured guidance through similarity criteria.
	KFLM	*Student difficulties:* Anticipation and partial identification of misconceptions related to the application of similarity criteria.
2	KoT	*Definitions:* Homothety as a transformation that scales figures from a center using a ratio (K). *Justifications:* Explanation of how the homothety ratio determines dilation and orientation. *Properties:* Preservation of shape and angles; orientation depends on the sign of K. *Representations:* Use of AR, GeR, and IR to communicate, visualize and reason about homothety. *Procedures:* Determining scale factors and applying them to transform figures. *Connections:* Links between homothety, proportionality, and similarity.
	KMT	*Teaching strategies:* Use of GeR and IR to visually communicate and distinguish direct and inverse homothety. *Guided questions:* Support for understanding the effects of positive and negative scale factors.
	KFLM	*Student difficulties:* Anticipation of errors in interpreting negative ratios and use of practical examples to scaffold conceptual understanding.
3	KoT	*Definitions:* TT as a statement about proportional segments induced by parallel lines intersected by a transversal. *Justifications:* Application of TT to support proportional relationships. *Properties:* Identification of corresponding segment ratios. *Representations:* Use of AR, NR, and GeR to model and calculate proportions. *Procedures: TT application to solve proportional problems.* *Connections: Integration of TT with broader concepts of similarity and dilation.*
	KMT	*Teaching strategies:* Coordination of GeR and IR to clarify spatial relationships in TT. *Guided questions:* Structured elicitation of student understanding around geometric configurations.
	KFLM	*Student difficulties:* Anticipation of confusion regarding the role of parallelism and the interpretation of segment ratios in visual tasks.

CONCLUSIONS

Grounded in the MTSK model and Duval's (1995) theory of registers of semiotic representations, this study aimed to characterize how a PST mobilized specialized knowledge through the coordination of semiotic registers across a sequence of geometry lessons on similarity, homothety, and TT.

Findings indicate that KoT and KMT were frequently mobilized across the three sessions, particularly through diagrams, algebraic expressions, NL, and gestures to support procedural understanding. These semiotic registers functioned as the main vehicles for communicating mathematical ideas and making the PST's specialized knowledge visible in classroom practice (e.g., Verdugo-Hernández & Caviedes, 2024). In lessons on similarity and homothety, the coordination of geometric and symbolic representations articulated proportional relationships and the mechanics of geometric transformation. However, some definitions lacked mathematical precision–for example, describing a proportion as "an equality between quantities", revealing limitations in KoT and a procedural orientation. These issues highlight the need for teacher education programs to strengthen definitional clarity (e.g., Caviedes et al., 2025) and promote greater conceptual depth (e.g., Seago et al., 2014; Tachie, 2020).

Evidence of KFLM was limited. Although the PST occasionally anticipated student difficulties (e.g., interpreting negative scale factors in homothety), there was minimal use of formative assessment or dialogic strategies to elicit and respond to student thinking. These findings align with prior research on the dominance of expository practices among PSTs and the disconnect between the use of multiple representations and the development of conceptual understanding (e.g., Iori, 2018; Seago et al., 2014). In the case of TT, for instance, geometric proportionality was addressed through ratios and diagrams but not linked to broader deductive or theoretical frameworks.

From a mathematics teacher education perspective, the findings highlight how initial preparation shapes classroom enactment. Although the PST had completed coursework on curriculum, common student difficulties, and mathematics education research, the lessons remained largely expository. This points to the need for teacher education to foster semiotic awareness (Duval, 2017; Iori, 2018; Presmeg, 2006) and the ability to engage with students' mathematical thinking–key elements of KFLM (Carrillo-Yáñez et al., 2018). As illustrated here, PSTs may evidence procedural competence without promoting exploratory discourse or reasoning, a limitation that might reflect the design of preparation programs themselves.

These findings must be understood within the context of a traditionally structured classroom with limited resources, where instruction is relied primarily on paper and pencil. This underscores the need to prepare teachers to use semiotic resources flexibly and meaningfully in such settings (e.g., Iori, 2018; Mwadzaangati, 2019, 2024). More broadly, the study illustrates how systemic constraints shape what pre-service teachers are able to enact (Bartell et al., 2017; Ensor, 2001). Future research should investigate how teacher knowledge and semiotic coordination evolve across varied practicum contexts and how reflective practices support the development of specialized knowledge. Longitudinal studies could also examine how early instructional patterns influence later teaching, particularly in relation to equitable access to conceptual mathematics.

While the study offers important insights, it is bounded by methodological limitations: the analysis focuses on a single case and context, observer effects cannot be fully excluded, and triangulation with student data or interviews was not feasible. Nevertheless, it provides a situated perspective on how teacher knowledge unfolds in real classrooms and highlights areas of growth for teacher training.

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Data availability: Data generated or analyzed during this study are available from the authors on request.

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