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Research Article

Conditions and constraints of implementing a mathematics lesson study-based PD program for Japanese pre-service teachers

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ABSTRACT

In this study, we aim to explore the conditions and constraints of implementing a lesson study-based professional development (LSPD) program for Japanese pre-service primary teachers. Based on the anthropological theory of the didactic–a well-established theoretical framework in mathematics education research–it offers three dimensions (institutional, educational, and personal) to frame and analyze the various conditions to implement the LSPD program and the constraints that impede them. We used a case study of an LSPD program for Japanese pre-service primary teachers, which offered pre-service teachers an opportunity to engage in mathematics lesson planning and participate in a lesson study conference. The results illustrated the conditions of the institutional context in which the LSPD program appeared, and the constraints in the difficulty of making the lesson study and PD coexist. We also found that the personal constraints on pre-service teachers' lesson preparation work were affected by the conditions and constraints in other dimensions (institutional or educational). Thus, this study contributes to a deeper understanding of the constraints that may shape or hinder teachers' practice and knowledge in the LSPD program at diverse but interrelated levels.

Keywords: anthropological theory of the didactic, ecology, lesson study, pre-service teacher education, professional development

INTRODUCTION

Lesson study is globally known as a powerful approach to teachers' professional development (PD) in the research field of mathematical education and beyond (e.g., Huang et al., 2019; Kim et al., 2021; Lewis & Lee, 2017; Quaresma et al., 2018). Chapman and An (2017) found that lesson study has also been integrated into PD programs to support pre-service and in-service teachers (e.g., Cajkler et al., 2014; Rick, 2011), allowing them to change their knowledge and disposition. This study focuses on a lesson study-based professional development program (hereafter, LSPD program) for undergraduate Japanese pre-service teachers (PTs) conducted in their final semester at a university. The final semester can be considered as the period of transitioning into a qualified teacher. In this phase, it is important for PTs to experience the professional practice of in-service teachers because novice teachers often experience a conflict between what they learn at university and what they practice at school (e.g., Ponte, 2017; Skott, 2001; Winsløw et al., 2009). This is the case for LSPD programs with initial teacher education (Larssen et al., 2018). For example, Winsløw et al. (2009) suggested the transition into the first years of teaching at three levels: epistemological, institutional, and personal. At the epistemological level, the transition appears in the teacher's knowledge of teaching, namely,

between the mathematical knowledge acquired in pre-service education and the knowledge adapted for the practical context. At the institutional level, the transition from university to school is crucial because one may experience different norms or cultural elements between them. At the personal level, one may need to change their capacity from a student in a community of students to a teacher in a professional community. These three levels are related and are deeply affected by PT education at university; thus, one may identify the affordances of the different levels that influence PTs' practice and knowledge during their final year of training at university.

Educational researchers (especially in Japan) are often involved as teacher educators (or 'knowledgeable others') in lesson study practice, naturally focusing on what is occurring during the program from an internal position (cf. Fujii, 2016). However, it is difficult for them to detach themselves from such internal positions and focus on the restricted aspects of lesson study (cf. Ponte, 2017), because the researchers are affected by certain contexts in which the lesson study is organized and implemented. This problem is also the case for the present study because the authors have played a role of teacher educators in an LSPD program. Therefore, it is worth investigating both the conditions and constraints that affect the implementation of an LSPD program (Bjuland & Mosvold, 2015; Karsenty, 2021; Ponte, 2017). For such attempts, researchers need to delineate a theoretical framework behind an LSPD program (Clivaz & Takahashi, 2018; Huang et al., 2019, Winsløw et al., 2018) that is detached from the internal position to gain a deeper understanding of the conditions, constraints, and challenges (cf. Bosch & Gascón, 2014). It may offer a reference model to explain what aspects of the LSPD program are seen as more or less successful. Thus, this study first describes the related literature on different forms of LSPD programs, and then, develops possible theoretical approaches to the reference model to identify and characterize the conditions and constraints. The essential aspects of the frameworks are then exemplified by illustrating the implementation of the LSPD program case study.

PD PROGRAMS FOR PTs VIA LESSON STUDY

Literature Review

Several studies have conducted a comprehensive review on lesson study in mathematics education (e.g., Clivaz & Takahashi, 2018; Huang & Shimizu, 2016; Huang et al., 2019) and a more focused review on lesson study in initial mathematics teacher education (or mathematics teacher preparation) (e.g., Bjuland, 2019; Ponte, 2017). Based on these reviews, the following four types of LSPD program settings can be identified:

- *University-based LSPD* programs that are organized as part of a PT education course (such as a method course) at a university (e.g., Chen & Zhang, 2019; Lewis, 2019).
- *School-based LSPD* programs that involve field-based experiences at schools with various adaptations of lesson study (e.g., Fernandez & Zillioz, 2011; Murata & Pothen, 2011).
- *Practicum-based LSPD* programs based on teaching practice at schools that are required in PT education courses (e.g., Nakamura, 2019; Rassmussen, 2016).
- *Project-based LSPD* programs that are conducted as part of a wider research project on lesson study (e.g., Bjuland & Mosvold, 2015; Clivaz & Miyakawa, 2020).

Several studies have also reported the challenges of the LSPD program's implementation, especially for pre-service education. Unlike the in-service settings, for example, the university-based LSPD programs for PTs, often require their participation in a 'top-down mandate' and 'high-stakes assessment' as their participation is graded by instructors (Lewis, 2019). Lewis (2019) reported that while PTs were collaboratively worked on curriculum recourses and lesson planning with the instructor, they barely contributed to the design of the research lesson. Chen and Zhang (2019) focused on improving PTs' lesson planning knowledge by placing lesson planning stages before and after a micro-teaching session in a method course. They found two common problems for PTs in both the initial and final lesson plans: 1) content analysis did not reveal the deep connections among all the concepts in the specified knowledge structure, and 2) the lesson plan emphasized the procedural knowledge over problem-solving knowledges.

According to Bjuland and Mosvold (2015), PTs often struggled to shift their focus from the organization of the teaching to pupils' learning, which was particularly challenging for lesson study implementation in pre-service settings.

Table 1. Different simplifications of lesson study for PTs in Japan

Full-cycle model (Fujii, 2016)	GS	LP	RL	PLD	R
Practicum-based lesson study for student teachers (Nakamura, 2019)		✓	✓	✓	
University-based LSPD program case study (Shinno & Yanagimoto, 2020)		✓	(✓)	(✓)	

Note. GS: Goal setting; LP: Lesson planning; RL: Research lesson; PLD: Post-lesson discussion; & R: Reflection

Therefore, as Ponte (2017) stated, it is important to acknowledge the dilemma. As PTs are not equivalent to experienced teachers or teacher educators, it is reasonable that 'they need some guidance and perhaps more structure than regular participants in lesson studies with practicing teachers' (Ponte, 2017, p. 178). Alternatively, to make an LSPD program successful, PTs 'need to have some freedom to make decisions and assume risks and need to see that their ideas are valued and considered (ibid.). This dilemma is also related to the problem of adaptation or simplification, which often depends on the context in which the LSPD program is implemented. The present study explains this issue from a Japanese context.

PD Programs via Lesson Study in Japan

In Japan, practicum-based LSPD is considered the most common and influential opportunity for PTs to initially experience lesson study (Elipane, 2012; Nakamura, 2019; Peterson, 2005). Nakamura (2019) explored the effect of lesson study on a PT during student teaching. Student teaching comprises three phases (lesson planning, research lesson, and post-lesson discussion), while the full-cycle model (Fujii, 2016) entails two additional stages (goal setting[1] and reflection) (see **Table 1**). Often, Japanese PTs, as student teachers, experience the three phases in a group, and work together both before and after the research lesson, even though one student teacher practices the lesson. Thus, most PTs may experience the three-phase cycle over several rounds during the practicum for three or four weeks, which seems relatively short and intense when compared to other countries.

Table 1 includes a university-based LSPD program case study (Shinno & Yanagimoto, 2020)–another common opportunity for Japanese PTs to participate in a lesson study conference organized by designated schools–typically known as university-affiliated schools (called *fuzoku* schools in Japanese). Some parts in parentheses indicate when PTs observe the research lesson and participate in the post-lesson discussion, rather than playing the role of a practicing teacher.

Miyakawa and Winsløw (2019) stated that lesson study conferences in Japan have been organized at national, prefecture, city, and local school levels[2]. A lesson study conference at a university-affiliated school is organized at the national or prefectural (regional) level. As university-affiliated schools have 'a leading role and offer various settings for teacher learning' (Miyakawa & Winsløw, 2019, p. 287), numerous teachers from different regions may attend to gain innovative ideas. Although most participants are in-service teachers, PTs at the university are also encouraged to participate.

As university professors (of mathematics education, for example) are often involved as the 'knowledgeable others' in lesson study conferences, it is reasonable to think that they consider this conference a good opportunity for PTs to learn about mathematics teaching in a professional context. However, few studies have been conducted on what and how PTs learn by participating in lesson study conferences. Such studies are under-represented because this opportunity is rarely coherent with a university's program (e.g., its method curriculum). Although participating in a lesson study conference can be valuable for PTs, it might be a 'special event' rather than part of a PD program.

Therefore, this study focuses on a university-based PD program designed to facilitate primary PTs' lesson planning skills through their participation in a lesson study conference organized by a university-affiliated school in Japan. Thus, to gain a deeper understanding of the conditions and constraints in the LSPD program's

[1] One might wonder why 'goal setting' is not included in the cycle for prospective teachers. 'Goal setting', by Fujii (2016), means that '[c]onsider long-term goals for student learning and development. Identify gaps between these long-term goals and current reality. Formulate the research theme' (p. 412). This usually appears in a school-based lesson study in Japan, but not in a practicum-based (or university-based) lesson study.

[2] The diversity of lesson study settings is also described in other literature (e.g., Baba et al., 2018; Isoda et al., 2007). Lesson study at the local school level is not typically considered a 'conference', but a school-based teacher training approach (called *kounai kensyuu*) (Stigler & Hiebert, 1999).

implementation, we address, and advance theoretical approaches based on different constructs of the anthropological theory of the didactic (hereafter, ATD). Accordingly, the research question is provided as follows: What are the conditions in the implementation of an LSPD in a university-based program and what are the constraints that hinder or prevent an efficient PD of PTs by using lesson study? By addressing this question from the perspective of ATD, we also discuss what we can learn from the constraints and challenges in the LSPD program for PTs in different dimensions.

THEORETICAL FRAMING

Anthropological Theory of the Didactic

This study considered ATD (Chevallard, 2019; Chevallard & Bosch, 2020a) as a grand theoretical framework to understand the conditions and constraints on the implementation of LSPD programs. ATD is a well-established theoretical framework for mathematics education and offers an 'ecological approach' to conditions and constraints that shape or impede practices and knowledge in an institution. Mathematics teacher education is a major research field for ATD, and many theoretical constructs have been developed from ATD (Bosch et al., 2020). This study drew on three main theoretical models (i.e., *paradidactic infrastructure*, *meta-didactic transposition*, and *didactic praxeology*) to frame essential factors that may affect the implementation of the LSPD for PTs in different dimensions. Although the three perspectives are individually oriented, they share at least two basic theoretical concepts in ATD, *praxeology,* and *infrastructure*.

The notion of praxeology is a primary construct in ATD, used to characterize any human activity (both practice and knowledge at stake) in an institution, considering the praxis (practical) and logos (theoretical) blocks (Chevallard & Bosch, 2020a). A praxeology consists of four components (**T**/**τ**/**θ**/**Θ**). In the praxis block (**T**/**τ**), *type of task* (**T**) indicates the problems of a given task, but *technique* (**τ**) is a way of performing the task. In the logos block (**θ**/**Θ**), *technology* (**θ**) is a form of explaining and justifying the technique, and theory (**Θ**) is employed to explain or justify the technology. Notably, the distinction between two elements in the logos block is often unclear in the praxeological analysis (cf. Winslow et al., 2018), therefore, we use **Λ** to denote '*technology* (**θ**) + *theory* (**Θ**)' for the characterization of the logos block. In the current study, praxeology is a theoretical concept that allows us to present a reference (praxeological) model (Barbé et al., 2005; Wijayanti & Winsløw, 2017).

Within ATD, infrastructure is a 'general concept: it refers to the underlying base needed to develop any determined, superstructural activity' (Chevallard & Bosch, 2020b, p. XXIX). Particularly, the didactic infrastructure refers to artefacts and resource systems that can be utilized for didactic practice, such as textbooks, backboards, ICT, and lesson plans (Winsløw, 2017). However, the notion of infrastructure in ATD is not reserved to the didactic ones but can be observed in various activities and systems that produce the conditions and constraints at different institutional levels. In this study, we referred to different infrastructures (i.e., para-didactic, meta-didactic, and didactic infrastructures) in relation to each theoretical perspective.

Three Theoretical Perspectives

Paradidactic infrastructure

During implementation of the LSPD program, institutional-level conditions or constraints can be crucial when PTs transition from university to school (Winsløw et al., 2009). For clarity, this study referred to the notion of *paradidactic infrastructure* (Miyakawa & Winsløw, 2013, 2019; Rasmussen, 2016; Winsløw, 2012). Although didactic infrastructure is concerned with teachers' practice inside the classroom, teachers often work outside the classroom before and after the lessons and participate in various meetings or PD programs. Miyakawa and Winsløw (2019) called such 'teaching-related' practices, *paradidactic activities*, and termed the given system of conditions and constraints for teachers' work outside classroom *paradidactic infrastructure*. Accordingly, lesson study is a typical example of *paradidactic system* (Miyakawa & Winsløw, 2019, p. 287).

Miyakawa and Winsløw (2013) also offered a model of paradidactic and didactic infrastructure that considered teachers' practice and knowledge inside and outside the classroom. However, this model primarily considers in-service teachers' activities; thus, some aspects need to be adapted to understand PTs' activities

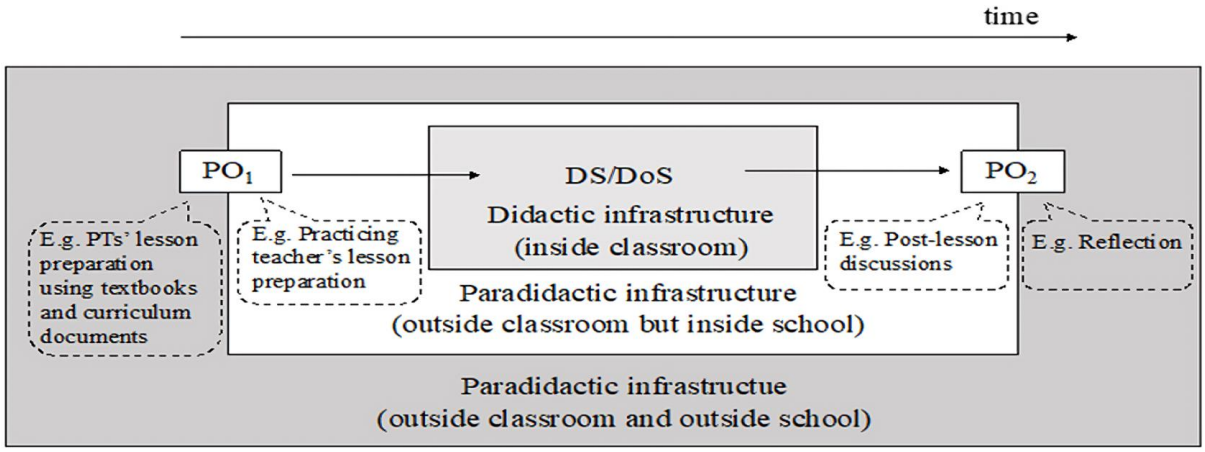


Figure 1. Paradidactic infrastructures inside and outside school (adapted from Miyakawa & Winsløw, 2013, p. 189), where PO: Paradidactic organization, DS: Didactic system, and DoS: Didactic observation system

outside of school (e.g., at university) during the university-based LSPD programs. Here, a 'school' refers to a venue where classroom teaching takes place, and the lesson study practice is implemented.

Figure 1 shows a model of 'dual' paradidactic infrastructures that considers both in and outside of the classroom and the school. Teachers' work outside the classroom is characterized as *paradidactic organization* (PO), comprising teachers' professional knowledge. **Figure 1** also indicates that the didactic infrastructure frames both the *didactic system* (DS) (where a teacher's didactic organization takes place) and *didactic observation system* (DoS) (where the observers' didactic organizations take place) (cf. Winsløw, 2012)[3]. This adapted model is especially applicable to PTs' paradidactic activities in the LSPD program because they develop their PO both when participating in lesson study at school and when engaging in preparation work (PO_1) and reflection regarding the lesson study at university (PO_2). Concerning PTs' lesson preparation (PO_1), they often develop their knowledge by referring to textbooks or curriculum resources. Regarding post-lesson discussions, PTs and lesson participants share the same PO_2, but the reflection at university is only associated with the PTs. They also develop their professional knowledge inside the classroom when observing a research lesson, even while not teaching. It means that lesson study (especially in the research lesson and the post-lesson discussion) shares the setting of the two kinds of paradidactic systems (LSPD program and lesson study conference) among the pre- and in-service teachers. This model enables the identification of some constraints and challenges in the LSPD program's implementation at the institutional level.

Meta-didactic transposition

The meta-didactic transposition (hereafter, MDT) model has been developed to understand the role of teacher educators or others (e.g., researchers) who may collaborate with teachers (Arzarello et al., 2014; Prodromou et al., 2018; Robutti, 2018). The 'didactic transposition', a theoretical basis of the MDT, which posits that certain mathematical knowledge exists in specific institutions, such as in a community of mathematicians, an educational system, a mathematics classroom, or a community of study (Bosch & Gascón, 2006). MDT is based on this notion and focuses on the process of mathematics teachers' professional learning.

> This framework is useful to describe a process–analogous to the didactical transposition–that occurs when a community of researchers work with a [group] of teachers in a PD activity. The term 'meta-didactical' refers to the fact that important issues related to the didactical transposition of knowledge are faced at a meta-level (Robutti, 2018, p. 4).

[3] Regarding the DS and the DoS, Winsløw (2012, p. 295) explained that a 'member of the lesson study group teachers the study lesson [research lesson] in her class, according to the lesson plan. The other members of the group observe the lesson, but usually do not intervene.'

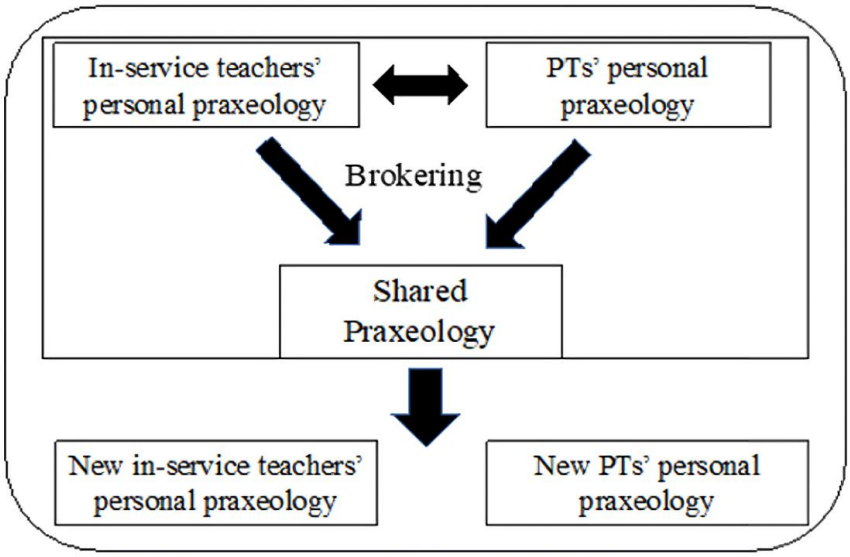


Figure 2. MDT model with adaptations from Arzarello et al. (2014, p. 355), where PTs: Pre-service teachers

Based on this tenet, we paid attention to two other features of the MDT model: namely, meta-didactic and didactic praxeologies. Meta-didactic praxeologies are the notion which describe researchers' or teacher educators' activities related to those of the teachers (Arzarello et al., 2014; Robutti, 2018).

Although the MDT model was originally used to guide and understand in-service teachers' professional growth via collaboration with researchers (or teacher educators), it can be used to understand the difference between capacities or positions (such as a practitioner or scholar) and the divergence between various persons from unique communities (Shinno & Yanagimoto, 2020).

We adapted this model to understand the conditions and constraints on PTs' praxeologies in relation to in-service teachers' praxeologies, and the evolution during the LSPD program (see **Figure 2**), in which researchers (or university instructors) may play a brokering role, facilitating the transition of mathematical concepts from one community another (cf. Robutti, 2018). Additionally, the meta-didactic infrastructure refers to artefacts and resources that can be used for meta-didactic activities or educational strategies in the context of PD, such as curricular documents, videos, and handouts, provided by instructors. Such a meta-didactic infrastructure may produce the conditions and constraints during the LSPD program.

From this model (**Figure 2**), this study investigates how a PT's initial personal praxeology (e.g., in their lesson planning) may change during or after the lesson study conference, and how they might be affected by the practicing (in-service) teacher's personal praxeology.

Didactic praxeology

Thus, the didactic praxeology is to model teachers' practice and knowledge. One can consider the distinctions between the praxis and logos blocks of knowledge and between the mathematical and didactic organizations of knowledge. *Mathematical organization* (MO) refers to mathematical practice and knowledge (mathematical praxeology), while *didactic organization* (DO) refers to teachers' pedagogies of MO. Although these notions are mostly used to characterize teachers' classroom teaching, they can also be employed to understand teachers' lesson design (e.g., Shinno & Mizoguchi, 2021) and the post-lesson discussion (e.g., Rasmussen, 2016; Rasmussen et al., 2020).

Regarding the Japanese lesson study, it is especially important to consider teachers' preparation work with *kyouzai kennkyuu*–the study of teaching materials–which is a crucial opportunity to develop teachers' professional knowledge (e.g., Fujii, 2018; Melville & Corey, 2021; Watanabe et al., 2008). In a *kyouzai kenkyuu* practice during lesson study, teachers' lesson planning can be characterized considering the didactic praxeology (including both MO and DO), which explains how curricular and textbook constraints can affect this planning phase (Shinno & Mizoguchi, 2021).

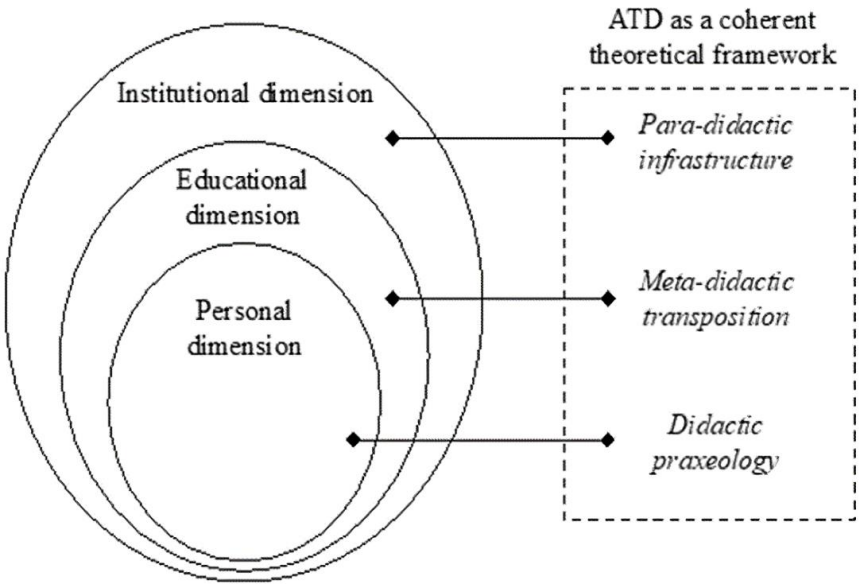


Figure 3. The theoretical approaches to multi-dimensional conditions and constraints (Source: Authors)

Multi-Dimensional Conditions and Constraints to be Considered

The three theoretical perspectives may shed light on the condition and constraints of the implementation of the LSPD for PTs from different dimensions, corresponding to each perspective. Our study conceptualizes each dimension as follows[4].

- *Institutional dimension* concerns the conditions and constraints on the para-didactic infrastructures of the PD system and community where the PD program is organized and implemented.
- *Educational dimension* concerns the conditions and constraints on meta-didactic transposition in which the level of the teacher educational process and the roles played by 'others' who are involved in the lesson study and the PD program (e.g., in-service teachers or university professors).
- *Personal dimension* concerns the conditions and constraints on PTs' didactic praxeologies that include mathematical and pedagogical knowledge for teaching, and the theoretical (logos) and practical (praxis) aspects of each knowledge type.

The three dimensions seem to be interrelated: the personal-level conditions and constraints are affected by those at the educational level, which are then affected by those at the institutional-level. According to Chevallard and Bosch (2020b), a '*constraint* is any condition which appears to be unmodifiable by occupants–acting as such–of a given institutional *position*' (p. XX-XXI). For our study, at the institutional level, the conditions and constraints are unmodifiable by any member of the lesson study group or by the LSPD program (including university instructors). Those at the educational level can be modified by the university instructors but not by the participating PTs or the practicing in-service teachers. As personal constraints rely on the conditions of the PTs' individual knowledge, a constraint may shape or hinder some PTs' pedagogical and mathematical knowledge (Ball et al., 2008), although not all PTs share the same conditions nor constraints. Thus, the three dimensions provide different units of analysis.

Figure 3 indicates our theoretical approaches to multi-dimensional conditions and constrains considered in this study. To aptly fit our theoretical approaches, the research question is reformulated as follows: what are the conditions in the implementation of an LSPD in a university-based program, and what are the constraints that hinder or prevent an efficient PD of PTs by using lesson study *at institutional, educational, and personal levels*?

[4] The three dimensions (institutional, educational, and personal) are similar but not identical to what Winsløw et al. (2009) referred to as institutional, personal, and epistemological levels for the initial teacher education. Especially, we reconceptualised the latter two levels as the educational and personal dimensions using the meta-didactic transposition and didactic praxeology.

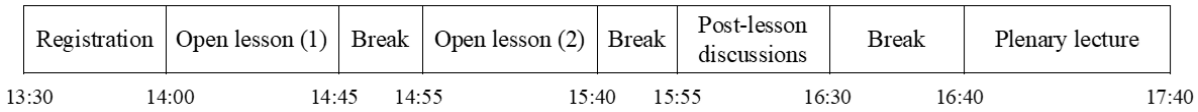


Figure 4. Lesson study conference timetable (Source: A booklet of the lesson study conference, translated by the authors)

CONTEXT AND METHOD

University-Based LSPD program

To exemplify the three main theoretical constructs based on the ATD, we drew on a case study of a university-based LSPD program for PTs in Japan. The case study involved a lesson study conference in a university-affiliated school where in-service and PTs participated (usually voluntarily). Although some results of this program were found in an earlier study (Shinno & Yanagimoto, 2020), the case study served as an example to identify the various conditions and constraints in terms of broader theoretical approaches.

The case study was rooted in a semester-long educational program for PTs at a university in Japan. PTs enrolled in the university's faculty of education to receive four years of training. The educational program, titled 'Practical seminars for the teaching profession', was taken by the PTs in their fourth (and final) undergraduate year. It offered opportunities to identify and reflect on weaknesses regarding the professional knowledge and skills needed for teaching and helped students to strengthen these aspects. The program consisted of two courses (I and II) that included various topics on PD. It was compulsory to take one seminar from courses I and II. The LSPD program was included as part of course I and was titled 'Designing a Primary School Mathematics Lesson'. Eight PTs participated in this program, having previously completed their teaching practice (practicum). They were from different major courses at the university; mathematics (1), art (1), social studies (2), and educational science (4).

The university instructors organized and implemented a three-week program, including an opportunity of the lesson study conference, which took place at an annual, half-day event held over two days at university-affiliated schools in October 2018 (one half-day was for primary schools and another for secondary schools). All 'open lessons' in the conference were centered on mathematics (five lessons for primary, three for lower secondary, and one for upper secondary classrooms).

Figure 4 shows the timetable of the lesson study conference in the primary school. The research lesson described in this study was included as the 'open lesson (1)'. Generally, university professors (or teacher educators) act as 'knowledgeable others' in the lesson study conference. During the research lesson the post-lesson discussion in which PTs observed and participated, a university professor took the role of a 'knowledgeable other' but was not responsible for the LSPD program[5].

While a general lesson study cycle consists of (1) goal setting, (2) lesson planning, (3) research lesson, (4) post-lesson discussion, and (5) reflection (Fujii 2016), the LSPD program for PTs in this study consisted of lesson planning, (observing) research lesson, post-lesson discussion, and reflection. This is similar to the practicum-based LSPD program reported by Nakamura (2019) (see **Table 1**). The participants (PTs) engaged in the lesson planning at university or at home, observed the research lesson, attended the post-lesson discussion at school, and had a reflective discussion at university.

During the program, three kinds of materials were collected from the eight PTs:

(1) a rough draft of the lesson plan ('pre-lesson preparation'),

(2) written notes from their observations of the lesson and the post-lesson discussion, and

(3) a lesson plan for the next lesson ('post-lesson preparation').

[5] The authors were also involved as 'knowledgeable others' during other open lessons and the post-lesson discussions. Thus, we were not in an internal position, allowing us to facilitate the PTs' activities during the conference.

These were used to assess the PTs' PD before and after the lesson study conference. This study also referred to the lesson plan written by the practicing in-service teacher and the textbook pages related to the lesson.

Reference Models to Analyze PTs' Lesson Preparations

The ATD perspective offers an explicit description of the praxeological elements that we use as a reference model to analyze PTs' lesson plans. For analyzing PTs and in-service teachers' 'initial personal praxeology' as well as 'new personal praxeology' (see **Figure 2** for MDT), we adopt the elements (**T/τ/θ/Θ**) as a reference model of their personal praxeologies. Their descriptions are as follows (see also **Appendix A** as a sample of the lesson plan):

- **T**: Didactic task described as the lesson goal in the lesson plan.
- **τ**: Didactic technique to perform and accomplish the lesson goal (often described as 'students' learning activities' in the section of 'lesson process').
- **θ**: Didactic technology to explain and justify the technique (often described in the section 'rationale for the teaching unit' in the plan).
- **Θ**: Didactic theory to justify and underpin the technology (often implicit but sometimes described in the 'rationale for the teaching unit' in the plan).

For example, regarding PT's initial praxeology, preparing a lesson plan to meet the lesson goal is a task type in the LSPD program. As already mentioned, if the distinction between **θ** and **Θ** is implicit, we use **Λ** to denote their knowledge (logos block). While analyzing the PTs' personal praxeologies, we also considered the *visibility* of the logos blocks (Wozniak, 2012). Referring to Barquero et al. (2019), a praxeology whose logos block is invisible is called a 'mute' praxeology, whereas a praxeology whose logos block is visible is called 'sound' praxeology.

A reference model is also utilized to analyze mathematics textbooks (Takeuchi & Shinno, 2020; Wijayanti & Winsløw, 2017). To analyze the MO in the textbooks which may affect PTs' lesson planning, we adopt the following elements:

- **T**: Mathematical task described as a word problem in the textbook.
- **τ**: Mathematical technique to solve the problem given in the textbook.
- **θ**: Mathematical technology to explain and justify the technique (often provided as a 'summary' or 'conclusion' highlighted in the textbook).
- **Θ**: Mathematical theory to justify and underpin the technology (sometimes implicit in the textbook).

The target knowledge in the MO for this study is associative law of multiplication, which is a property of algebraic operation. Although a theoretical discourse is rather implicit in the primary mathematics textbooks, it can be explained, as follows:

> 'For the addition and multiplication of numbers, associativity is expressed by the following identities: $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c$.
>
> A general binary operation ★ is associative (or, which is the same thing, satisfies the law of associativity) if the identity $a★(b★c) = (a★b) ★c$ is valid in the given algebraic system' (European Mathematical Society, n. d.).

In our praxeological analysis, particularly when considering educational and personal conditions and constraints[6], we use above two reference models to illustrate (para)didactic and mathematical praxeologies in PTs' development during the program.

[6] For the institutional dimension, the Japanese paradidactic infrastructures presented by Miyakawa and Winsløw (2013, 2019) and **Figure 1** in this paper may work as a reference model to understand teachers' paradidactic activities. However, we do not intend to apply it to a praxeological analysis for the present study.

THE ILLUSTRATIVE CASE STUDY

This section exemplifies the theoretical approaches via the case study, and the conditions and constraints are described on the institutional, educational, and personal dimensions in terms of paradidactic infrastructure, MDT, and didactic praxeology.

Institutional Conditions and Constraints

Didactic and paradidactic infrastructures

In the case study, the paradidactic system was a prefecture-level lesson study conference held in a university-affiliated school, sponsored by local education boards. The Japanese paradidactic infrastructure constitutes conditions on the lesson study group and the implementation of the LSPD program. The conference was organized and conducted by primary and secondary mathematics teachers from three university-affiliated schools[7] and university professors of mathematics education. One school was selected as the conference venue, and the teachers from the other two schools visited the 'host' school to collaborate in the lesson study group. This institutional setting led to the following constraints on the lesson study group.

First, each university-affiliated school had its own goal setting for lesson study within the school, while the lesson study group also had goal setting for the conference. Although all members of the lesson study group discussed their goals before the conference, it was sometimes difficult to design research lessons according to their goals, and some teachers struggled to match the goals of their schools with those of the lesson study conference. Accordingly, the goal setting for the lesson study conference was often based on the general objectives of the Japanese national curriculum, which was an overarching theme and was easy to share within the lesson study group.

Second, if the teacher who taught the research lesson was from another school, they may not have known much about what and how the students had learnt before the particular lesson. This constraint may have affected the teacher's lesson planning, owing to challenges in designing a lesson that was coherently planned with the textbook unit; they usually taught only one lesson for the lesson study, rather than a series of lessons throughout the unit. Therefore, teachers' often plan a 'topical' lesson on an isolated topic, unconnected to the unit that the class students are learning at that time. But, this situation may cause challenges regarding how the lesson study can connect with the class students' daily learning, based on the curriculum or textbooks.

Third, the format of the lesson plans in the conference was based on the format used in the teachers' schools rather than a common format in the lesson study group. Although the different formats of the lesson plans may have lacked commonality within the community, it is also possible that the lesson plans could be affected by the specificities of each 'school culture'. Therefore, while it might have been a constraint, it was not a challenge for the practicing teachers in this group. Unlike the teachers in the organizing group, most participants (who were mostly in-service teachers) in the conference did not have an opportunity to know about the culture or tradition of the different schools. For such people, the variations sometimes may have constituted as a constraint and hindered the consistency in the lesson plans when they observed different research lessons (as they could attend two different 'open lessons' in the conference).

PTs' didactic observation system

According to PTs who participated in the conference, DoS had some constraints due to the assignment and assessment of the LSPD program. The PTs had engaged in lesson planning before joining the conference. During their observation of the research lessons and participation in the post-lesson discussion, the PTs were required to take notes in a specified format (see **Figure 5** for a sample). To concentrate on the observation while taking notes[8], the lesson plan of the research lesson was not given to the PTs, although it was provided

[7] In the case study, there were three different affiliated schools with a university in one prefecture. Each school had three mathematics teachers at primary and lower secondary schools, and five for the upper secondary level.

[8] This format (**Figure 5**) comprised two columns–'What you noticed about the mathematics subject matter' and 'What you noticed about the pedagogical acts'–completed during the lesson observation. It also required PTs to summarize the teacher's comments, the participants' questions and comments, and their own questions and comments, which were completed during the post-lesson discussion.

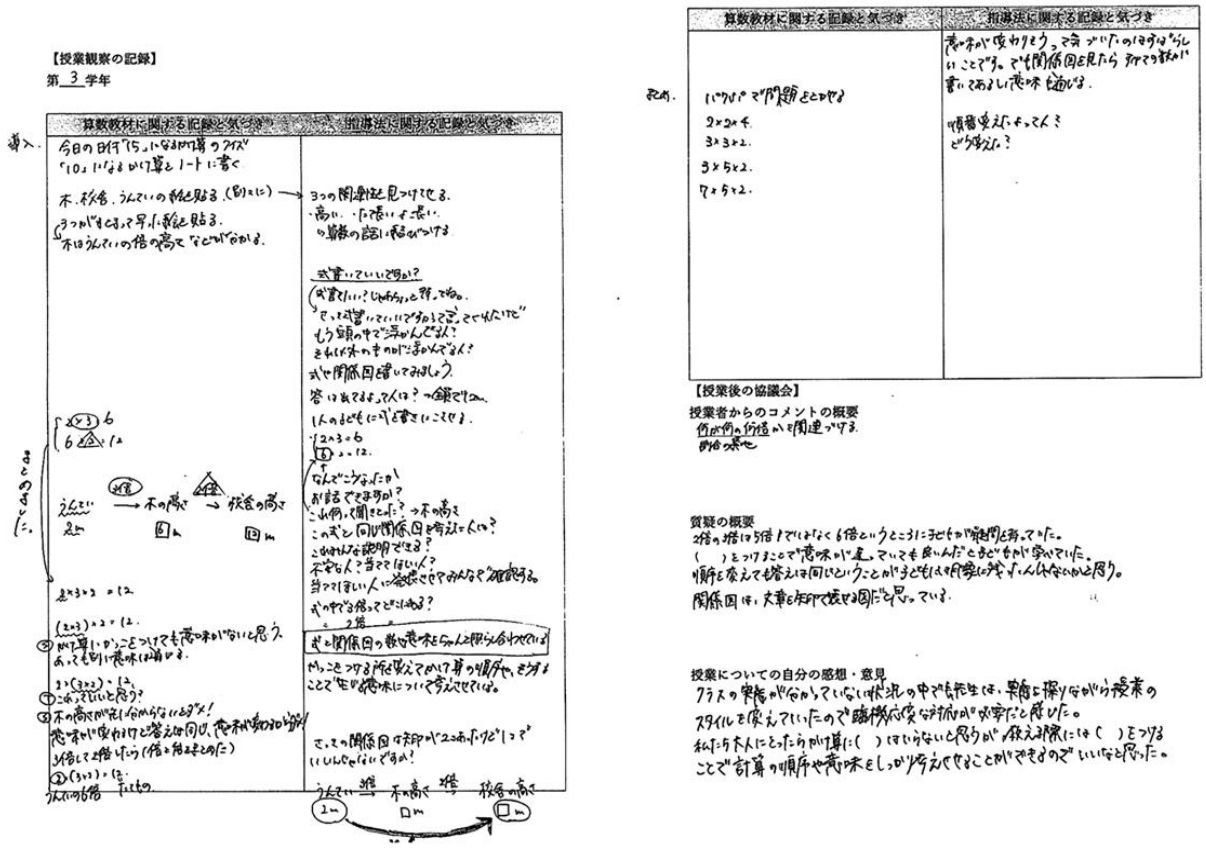


Figure 5. Production of the lesson observation by a pre-service teacher (Source: Authors' own elaboration)

after the conference for reflection at university. During the observation, this might have caused them difficulty in distinguishing between 'what is intended in the lesson plan' and 'what is actually done in the classroom'. Although it is unusual to participate in a lesson study without having the lesson plan, this was necessary for the instructors to assess what they had observed from the LSPD program. However, this constraint did not lead to obstacles because some PTs tried to listen to and take notes about what the teacher had commented regarding their intentions in the post-lesson discussion.

Educational Conditions and Constraints

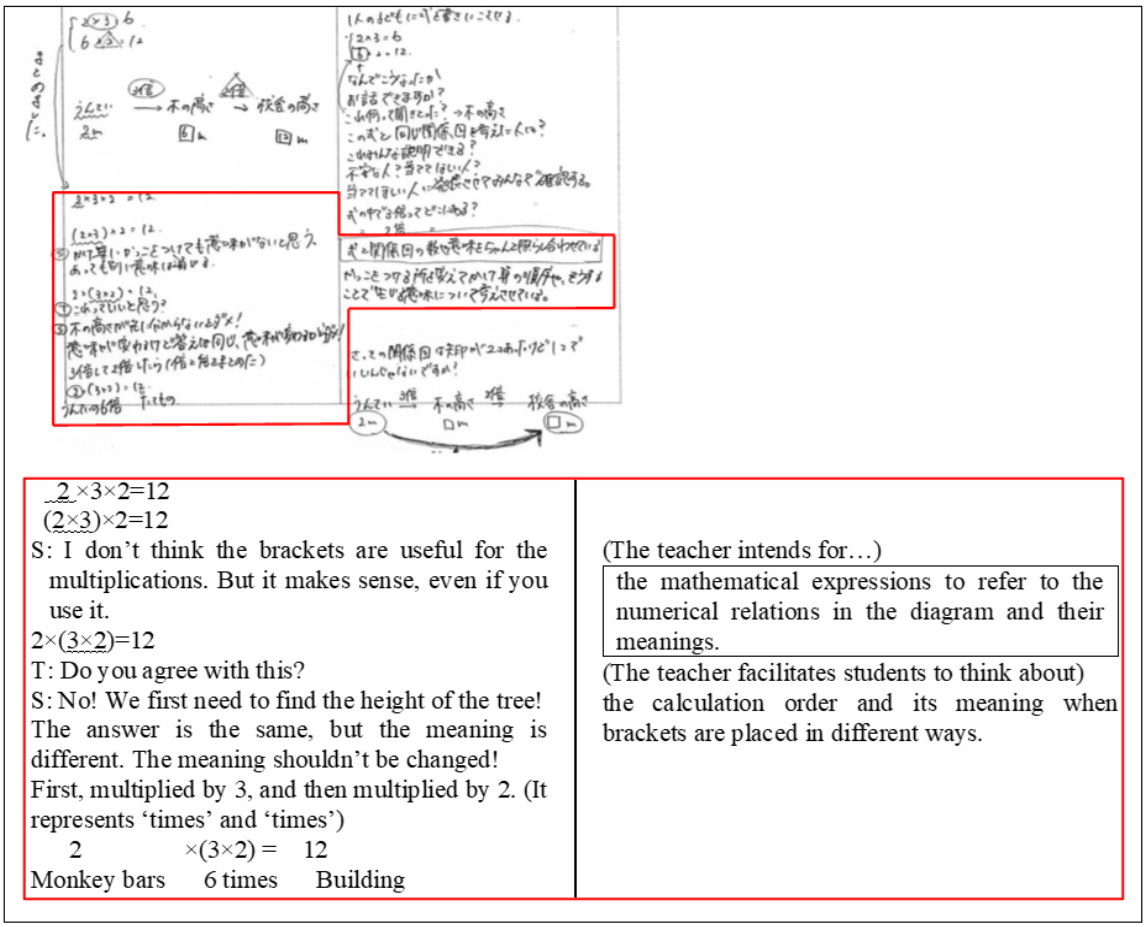
Before the PTs joined the conference in the LSPD program, they were assigned a lesson planning task. The instructors asked them to draft a lesson plan for grade 3 students by referring to the curriculum and textbooks. Although lesson plans were already obtained from the members of the lesson study group, this study only provided the lesson goal and the word problem used therein, which were excerpted from a research lesson plan (see **Appendix A**; translated into English by the authors). The presented task (word problem) was titled 'calculation order' (situated in two pages of the textbook used in the classroom), and was intended to introduce the associative law of multiplication, as follows:

- The lesson goal: Through activities such as representing two quantitative relations using diagrams, students enable to think about the idea of multiples and understand the associative law.
- The word problem: Compare the heights of the following monkey bars, the tree, and the school building. The monkey bars are 2 m high. The tree is three times taller than the monkey bars, and the school building is twice as tall as the tree. How tall is the school building?

One of the most important meta-didactic stakes of the LSPD program was that it allows PTs to consider the relation between the goals of the lesson (as a didactic stake) and the didactic techniques carefully (e.g., exploring the textbook though *kyouzai-kenkyuu*; also see the 'Didactic praxeology' section). Therefore, creating such an opportunity for PTs to work on lesson planning though *kyouzai-kenkyuu* was a meta-didactic technique that was led by the instructors (the authors as teacher educators). Another important meta-didactic technique was to relate the PTs' lesson planning activities to the lesson study conference (as a paradidactic system). This

Table 2. Initial personal praxeologies

PT (A)	Practicing (in-service) teacher (B)
$\mathbf{T}_{A\&B}$: To design a lesson for students to understand the associative law of multiplication.	
$\boldsymbol{\tau}_A$: Using the textbook approach as is $\boldsymbol{\Lambda}_A$: (not specified)	$\boldsymbol{\tau}_{B1}$: Comparing the two expressions (2×3)×2 and 2×(3×2) using diagrams. $\boldsymbol{\tau}_{B2}$: Overcoming possible misconceptions by students, such as '3 times by 2 times' is '5 times'. $\boldsymbol{\Lambda}_B$: The lesson provides a basis for understanding the idea of multiples and proportions that must be taught to higher grades.



2×3×2=12
(2×3)×2=12
S: I don't think the brackets are useful for the multiplications. But it makes sense, even if you use it.
2×(3×2)=12
T: Do you agree with this?
S: No! We first need to find the height of the tree! The answer is the same, but the meaning is different. The meaning shouldn't be changed! First, multiplied by 3, and then multiplied by 2. (It represents 'times' and 'times')
2 ×(3×2) = 12
Monkey bars 6 times Building

(The teacher intends for…)
the mathematical expressions to refer to the numerical relations in the diagram and their meanings.
(The teacher facilitates students to think about) the calculation order and its meaning when brackets are placed in different ways.

Figure 6. Notes taken by the PT during the lesson observation (Source: Authors' own elaboration)

was possible during the actual research lesson (as a didactic system or didactic observation system) which allowed the PTs to compare their first lesson plan and its realization in the classroom. This opportunity may have allowed them to learn from the practicing teacher's lesson planning (as well as the teacher's didactic activities) by observing the research lesson and attending the post-lesson discussion, which may, then, lead to a change in their personal didactic praxeologies.

However, one educational constraint in the LSPD program (as a meta-didactic system) was that there was no 'direct' collaboration between the PTs and practicing teachers during the teacher educational process, both before and after the lesson study conference. For example, while planning the lessons, the PTs and practicing teachers worked independently, and completed their first plan without any sharing any information before the lesson study conference. This is because the PD program and the lesson study conference exist as two distinct systems, although the teacher educators tried to play a brokering role between them.

Initial personal praxeologies

In terms of the MDT model (see **Figure 2**) and our reference model, **Table 2** presents a PT's initial personal praxeology and an in-service practicing teacher's personal praxeology, as evidenced from their lesson plans. Both praxeologies shared the same task (**T**). To scrutinize the practicing teacher's praxeology, the techniques and logos block can be interpreted based on the lesson plan shown in **Appendix A**.

Table 3. Shared and new personal praxeologies

Shared praxeology in the lesson study (C)	$\mathbf{T_C}$: To design a lesson for students to understand the associative law of multiplication. $\boldsymbol{\tau_C}$: How to respond to students' different sense-making processes regarding the two expressions. $\boldsymbol{\Lambda_C}$: Students tend to understand expressions with 'meanings' by relying on the word problem.
PT's new praxeology (D)	$\mathbf{T_{D1}}$: To design a lesson for students to understand and apply the associative law of multiplication. $\boldsymbol{\tau_{D1}}$: Focusing on the calculation order to make the calculation easier. $\boldsymbol{\tau_{D2}}$: Considering multiplications with bigger numbers (e.g., multiplied by 15). $\boldsymbol{\Lambda_D}$: This lesson is meant to apply the associative law to promote mathematical thinking skills.

Figure 6 shows what the PT noticed during the lesson–the students' different sense-making of the two mathematical expressions ('2×3×2=12' and '2×(3×2)=12')–with only one part translated into English (i.e., framed) from their notes using the specified format (see also **Figure 4**). It shows the differences regarding both the pre-service and in-service teachers' expectations of the lesson.

Shared and new personal praxeologies

Table 3 indicates the shared praxeology by the group of teachers (the group of PTs) and the PT's new personal praxeology. The shared praxeology was affected by at least three kinds of experiences: lesson observation (**Figure 6**), post-lesson discussion, and reflective discussion of the lesson study at university.

Particularly, the university instructors attempted to make the shared praxeology as explicit as possible in the reflective discussion. For example, the use of observation sheets from some participants (e.g., **Figure 4**) enabled them to express what they had noticed and learnt from the lesson study[9]. The technique ($\boldsymbol{\tau_C}$) and logos ($\boldsymbol{\Lambda_C}$) in **Table 3** are based on such discussions. However, the shared praxeology was somewhat unclear regarding the in-service teacher's initial praxeology in **Table 2**, as the logos part included 'this lesson provides a basis for understanding the idea of multiples and proportions that must be taught to higher grades'; however, this did not appear in the shared praxeology. Owing to the constraints of the program, this study could not invite the practicing in-service teacher to the reflective discussion at the university. The university instructors could only play a brokering role between different praxeologies in the community of PTs.

This study's analysis observed the change in the PT's praxeology but could not determine whether the practicing in-service teacher's praxeology changed through the lesson study. Considering the MDT model, the interactions between the PT and in-service teachers' praxeologies were not explicit. This is because the in-service teachers were members of the lesson study group in the conference but were not participants of the LSPD program.

Personal Conditions and Constraints

The personal conditions and constraints were conceived as those of the teacher's didactic praxeologies, which were heavily affected by the conditions and constraints of the other two dimensions. However, some constraints were also associated with PTs' personal background of PD course at the university. For example, only one participant was from mathematics course, but others were from different major courses. It is evident that this participant was more advanced in mathematical knowledge, which may affect his activities in the LSPD program. Therefore, we focused on other seven PTs' didactic praxeologies.

Regarding the PTs' mathematical organization (MO) in the pre-lesson preparation, there were commonalities with the textbook's MO. **Figure 7** and **Figure 8** show the textbook pages (translated into English) used for lesson planning. Through our reference model, the four praxeological components can be interpreted as follows (note that for the two techniques ($\boldsymbol{\tau 1}$, $\boldsymbol{\tau 2}$) below, two different diagrams in the textbooks may also be included accordingly[10]):

– **T**: The word problem (see **Figure 7**).

[9] Therefore, the 'shared praxeology in the lesson study' in **Table 3** is considered as a transition between the initial and the new praxeology in PTs' lesson planning. Owing to this characteristic, our reference model does not work well to identify the 'shared praxeology'. As the model is presented to analyze lesson plans, the 'shared praxeology' is interpreted from the observation sheets.

[10] In this textbook series, this diagram (called the 'relational diagram' [*kankei-zu* in Japanese]) is particularly emphasized in different chapters.

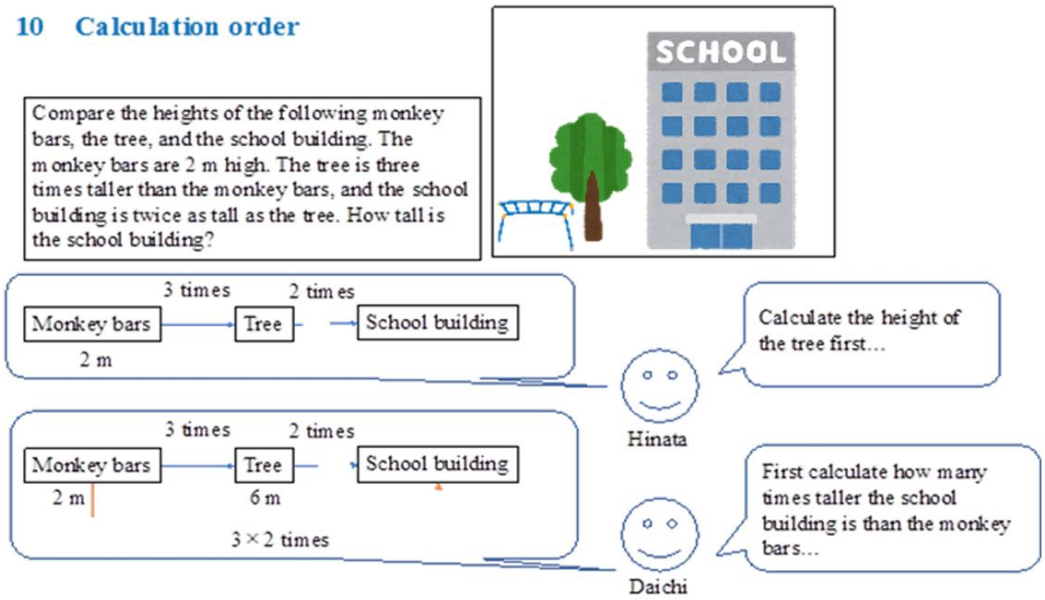


Figure 7. Textbook pages (reconstructed from Shimizu & Funakoshi, 2012, p. 16).

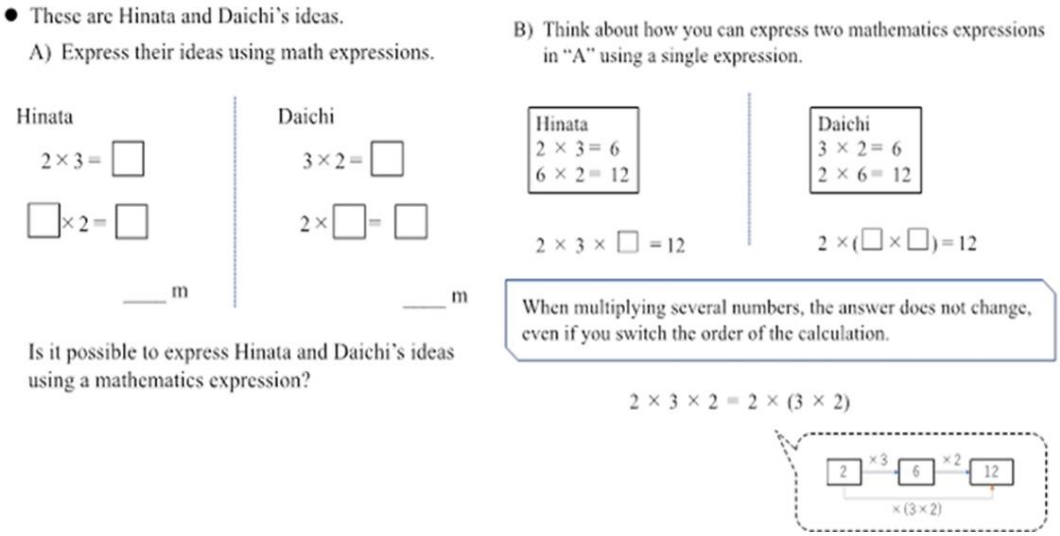


Figure 8. Textbook pages (reconstructed from Shimizu & Funakoshi, 2012, p. 16-17).

- **τ1**: Hinata's idea for the calculation: '2×3=6, 6×2=12'.
- **τ2**: Daichi's idea for the calculation: '3×2=6, 2×6=12'.
- **θ**: When multiplying several numbers, the answer does not change, even if one switches the order of the calculation (see **Figure 8**).
- **Θ**: Associative law of multiplication.

Regarding the PTs' didactic praxeologies (see also **Table 2**), their lesson plans were almost 'copied-and-pasted' from the textbook's approach and were rather straightforward. For example, 1) the word problem is presented, 2) two ideas come up, compare them, 3) think about how to combine two expressions into one expression, and finally, 4) summarize by considering '2×3×2=2×(3×2)'.

For example, the technique (τ_A) is equivalent to the mathematical techniques (**τ1** and **τ2**) in the textbook, but it has almost no additional didactic input. However, the techniques (τ_{B1}, τ_{B2}) in the practicing teachers' praxeologies (also see **Table 2**) are didactic ones which are more than the textbook's approach. By comparing the logos blocks in their praxeologies (Λ_B and Λ_B in **Table 2**), the PTs' personal praxeologies (A) were found to be 'mute' praxeologies in which the logos blocks are invisible (while that of the in-service teachers are visible).

Some PTs improved their post-lesson preparation while some did not change, even after the reflective discussion at the university. As shown in **Table 3**, an educational condition on the meta-didactic activities led to a successful case, indicating that a set of components in a PT's new personal praxeology could be identified by analyzing their lesson plan for the 'next' lesson, which featured a new task design. For example,

> 'Compare the weight of a cat, a dog, and a bear. The cat weighs 2 kg. The dog is three times heavier than the cat, and the bear is 15 times heavier than the dog. What is the weight of the bear?'

By using this word problem, it seems that the teacher intended for students to calculate '2×3×15' by breaking 15 down into '5×3', as follows: 2×3×15=2×3×(5×3)=(2×5)×3×3=10×3×3=10×9=90.

According to this PT's lesson plan, the associative and commutative laws could be applied to the calculation to make it easier by including a factor of '10'. The techniques ($\boldsymbol{\tau_{D1}}$, $\boldsymbol{\tau_{D2}}$) and the logos ($\boldsymbol{\Lambda_D}$) explained the main focus of the lesson plan that they developed. This change in their logos block can be considered as the evolution from 'mute' to 'sound' praxeologies (Barquero et al., 2019). Thus, the task design allowed students to undergo the abstraction process from the word problem to the expressions, and to focus on the order of the calculation[11].

Conversely, the same didactic milieu led some PTs to be less successful because they did not improve the task design or process, even in their post-lesson preparation. For example, one PT's lesson plan was based on the in-service teacher's lesson plan, with almost no additional input, despite their understanding that this lesson plan was for the next lesson. In their lesson plan, a new task was designed, but it was not significantly different from the task in the research lesson:

> 'Compare the heights of the pencil, chair, and desk. The pencil is of 9 cm in height. The chair is five times taller than the pencil, and the desk is twice as tall as the chair. How tall is the desk?'

Some sentences in the PT's lesson plan were highly similar to those in the in-service teacher's lesson plan (e.g., noticing student misconceptions). It seems that the PT could understand what they did not consider in their pre-lesson preparation but could not apply or adapt these issues into new preparation work. Therefore, this study interpreted that the PT's didactic praxeology developed into a shared praxeology (see **Table 3**) but did not change into a new praxeology in the post-lesson preparation. There were also some PTs whose post-lesson preparations relied on the textbook's (exercise) task without major adaptation, implying that their lesson observations and reflective discussions were ineffective in their development of lesson planning knowledge. Such results can be explained by PTs' personal constrains, such as the invisible nature of the logos block in their personal praxeologies in the LSPD program.

DISCUSSION AND CONCLUSION

Responses to the Research Question

Compared with the LSPD for in-service teachers, more complex affordances of the different dimensions influence PTs' practice and knowledge in various settings. To better understand such complexities, this study explicated how different theoretical approaches with reference models can frame the conditions and constraints of the LSPD program's implementation for primary PTs in terms of the intuitional, educational, and personal dimensions. A Japanese case study was used to illustrate such multi-dimensional conditions and constraints, using different theoretical approaches (paradidactic infrastructure, MDT, and didactic praxeology). **Table 4** summarizes the conditions and constraints.

The study contributes to the literature by showing what can be learnt from the constraints and challenges. The adapted model of paradidactic infrastructure allowed us to identify some institutional constraints, such as the special conference setting for in-service teachers, and the assessment of lesson observation for PTs. The MDT model was also adapted to characterize how PTs' personal praxeologies were affected by the lesson

[11] Although this can be considered a successful aspect in the mastery of lesson planning skills, some aspects can be identified that have not yet been improved in this case, such as curricular content knowledge exhibited in the in-service teacher's lesson plan (see also **Appendix A**).

Table 4. Conditions and constraints found at different dimensions

Dimensions	Conditions	Constraints
Institutional	Para-didactic infrastructure in which a lesson study conference appears (e.g., led by a university affiliated school; a prefecture-level open conference, etc.)	Difficulty of making lesson study conference & PD program coexist (e.g., organizational constraints of lesson study group; limited resources for lesson observation)
Educational	Meta-didactic infrastructure in which the role played by the lesson study as an educational strategy (e.g., opportunity to prepare the same lesson as the lesson which PTs observe in the lesson study, etc.)	Inequivalent positions between the in-service teacher and PTs in the lesson study (e.g., the in-service teacher did the lesson, but PTs did not but observe it; no opportunity to collaborate with in-service teacher)
Personal	Didactic infrastructures for lesson planning, which include curricular documents, textbooks, & some information from observed research lesson & the post-lesson discussion	PTs' (initial) didactic praxeologies include its techniques that rely on textbook's mathematical techniques; but it is not developed didactically (e.g., anticipating students' thinking & using different representations of mathematical content)

study and to identify the educational constraints on the collaboration between the PTs and the practicing teacher. The notion of didactic praxeology explained how PTs' mathematical organizations were restricted to textbooks and the challenges they faced in adapting their lesson planning skills.

Accordingly, this study demonstrated that constraints appeared to be unmodifiable by individuals at certain levels (e.g., the personal dimension), but the personal conditions and constraints could be modified by someone who worked at the educational or institutional levels. Therefore, the conditions and constraints at the three dimensions affected each other, especially as the institutional constraints significantly affected the constraints at other levels. Thus, a theoretical contribution of this study, is illustrating how different theoretical constructs based on ATD can be adapted and combined for a deeper understanding of the phenomenon being studied[12].

Key Contributions and Implications

This study has two main contributions. First, it suggests the importance of investigating the conditions needed to establish successful PD proposals and the constraints that impede their development as normalized activities (cf. Bjuland & Mosvold, 2015). As the type of educational format–LSPD–is the object of divergent adaptations worldwide; the details provided about a Japanese case can aid in deciphering the specificities of lesson studies and the difficulties they can find in their original institutional setting. Second is the theoretical contribution. It is appropriate to adopt a broad perspective by delimiting a large unit of analysis to include the institutional settings instead of only considering what happens in the classrooms during their implementation.

For such a broad perspective, our study utilized three different perspectives with the praxeological reference models, which enabled an external perspective when researching within the ATD (Bosch & Gascón, 2014), along with highlighting some implicit cultural elements. The literature (e.g., Bjuland & Mosvold, 2015; Chen & Zhang, 2019; Lewis, 2019) has shown that LSPD for PTs face multifarious challenges; however, most studies have only focused on one particular dimension (e.g., concentrating more on the personal dimension to find the weakness of teachers' knowledge for teaching than the issues from other two dimensions). However, this study implies that the conditions and constraints in the LSPD program's implementation are multifaceted.

Another implication is that we can only modify the conditions and constraints at a specific level if we work at a different level (cf. Chevallard & Bosch, 2020b). Although the university instructors cannot change any condition at the institutional level, they can alter some conditions at the educational level. Therefore, it is worth investigating effects on the pre-service teachers' activities in the program when some conditions and constraints are modified at the educational level for improvement.

[12] While the three theoretical constructs are based on the ATD, to the best of our knowledge, the MDT model has been developed with little theoretical links with the paradidactic infrastructure.

Limitations and Perspective for Further Research

Further research is needed to address the following limitations. First, this study focused on a university-based LSPD program for Japanese primary PTs. Future research should consider the extent to which this approach is applicable to other LSPD programs for PTs or in-service teachers in different countries. Investigating different contexts based on this study is important to understand and compare commonalities and specificities. Second, this study paid considerable attention to the theoretical considerations by using one case study as an example; future research should address more empirical investigations. Third, the program setting did not allow the PTs to work collaboratively. However, lesson studies usually involve teacher collaboration in the professional community (e.g., Jaworski et al., 2017). Therefore, designing a collaborative setting for the LSPD program in future.

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Declaration of interest: Authors declare no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

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APPENDIX A

Mathematics lesson plan
Order of calculation:
To clarify "what is how many times of what" and to develop a basic understanding of proportions

1. Grade: Grade 3
2. Main subject: Order of calculation
3. Rationale for the teaching unit

"If I am told that ○ is □ times that of △, I do not know which to divide by which."
Several upper grade students have this kind of question. Thus, since many students are not good at proportions, we should improve the way we teach proportions. I think this is not so easy, although that is important as well.

In this teaching unit, students will examine multiplicative variables and connect them to the associative law of multiplication by considering "what is how many times of what". The teaching unit "How much does it increase/decrease" for Grade 2 and the unit "How many times is it?" for Grade 3 are the learning contents which the students have already learnt in their curriculum and these units then are connected to the teaching unit "How many times does it increase/decrease (solving by reverse thinking)" in Grade 4. The textbook chapters are well arranged to lead to the rules of calculation based on this learning.

The task of this lesson is to make students consider the "what" (i.e., the reference variable) of "what is how many times of what." If the task has only two elements, as it has in the past, the answer may be reached by simply doing the calculations without being aware of the reference variable. However, if the task involves three elements, as in this case, the students will have to be aware of the reference variable. For example, students are asked to focus on the reference variable and explain it, while providing the misconception "isn't □ times of △ times equal to (□ +△) times?" As mentioned earlier, this idea of the reference variable should be the basis for the learning of "proportions" in Grade 5. We also want students to consider the function of multiplying by ○, and to deepen their understanding of multiples by showing the function in a relationship diagram. By comparing and contrasting the formulae (2 x 4) x 2 and 2 x (4 x 2) at the end of the lesson and relating them to the diagram, the students will be able to connect them to the associative law of multiplication and further deepen their understanding of the meaning of multiplication.

It would be desirable to end the lesson with an activity that fosters a mathematical way of perceiving and thinking by having the students reflect on what they have learned and consider what kind of learning led to their learning in the lesson, rather than merely giving their impressions of the lesson.

(Omitted: items 3, 4, 5)

6. The aim of the lesson

Through activities such as representing two quantitative relations using diagrams, students enable to think about the idea of multiples and understand the associative law.

7. Plan for the lesson process

Learning process	Learning activities and content	Notes on teaching	Assessment perspectives
Prepare	7. Let their imaginations expand by examining the diagrams of trees, monkey bars, and school buildings.	○Let them express their imaginations freely while focusing on what is related to mathematics, so that they can understand the problem.	Be interested in problems and be willing to examine them. [Attitude]
Seek	2. Understand the problem. Compare the heights of the following monkey bars, the tree, and the school building. The monkey bars are 2 m high. The tree is three times taller than the monkey bars, and the school building is twice as tall as the tree. How tall is the school building?		
Develop	3. Recall the previous lesson and draw diagrams etc .	○ Clarify the relationship between multiples by using a tape diagram and a number line, so that students understand what is □ times of what. ○ Provide misconceptions such as "isn't 2 times of 3 times equal to 5 times?" to make them reconsider the relationship between multiples.	Can think about how many times the reference is multiplied and explain the relationship to the diagram. [Thought, judgement, representation]
	4. Express as one equation and compare them. 2 x 3 x 2 = 12 2 x (3 x 2) = 12	○ Clarify what was used to find the height of the school building by limiting to a single expression. ○ Relate to the rules of calculation (associative law of multiplication).	Can understand the associative laws. [Knowledge, skills]
Deepen	5. Consider application problems.	○Ask students to identify the function of multiples by using diagrams and relationship diagrams.	
Reflect	6. Reflect on each class.	○Help students develop a mathematical way of thinking by making them aware of the continuity of learning and allowing them to verbalise their reflections.	

(Source: A booklet of the lesson study conference, translated by the authors)