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Activating pre-service mathematics teachers' critical thinking

Applebaum, Mark
Kaye Academic College of Education, Beer Sheva, Israel
For correspondence: mark@kaye.ac.il

Abstract:
Teachers' critical thinking skills are essential for fostering the development of the same skills in their students. To demonstrate how teachers' ability to examine solutions critically can be developed and supported, we analyse a classroom activity performed bya group of pre-service secondary school mathematics teachers(N=37)who were asked: (1) to solve a geometrical problem; and (2) to reflect upon different strategies that lead to contradictory solutions. The data demonstrate difficulties in students' ability to analyse possible mistakes and cope with mental dissonance when faced with contradiction.

Keywords: Critical thinking, problem solving, geometry, pre-service teacher training

Introduction

Critical thinking skills are an important part of educating people to be capable of meeting challenges and solving complex problems in the 21stcentury (Bailin and Siegel, 2003).Solving such problems requires skills, such as understanding the problem, analyzing the components of the problem and their roles, choosing a strategy and decision-making, realising the plan and taking a critical view back for revising the results and building a new improved plan. In addition to all the components mentioned above (which generally fit Polya's plan for solving problems), nowadays solving complex problems includes understanding interdisciplinary and multidisciplinary connections, applying the results to real life and combining the use of media tools. One of the main goals of education is preparing students for life in the future and solving different and sometimes non-standard complex problems. School mathematics plays an important role in learners' practice in solving complex problems, there by preparing students for life, in general, and developing their critical thinking skills, in particular.
Mathematical problem solving is known as a potentially rich activity conducive to developing all of the above mentioned skills (NCTM, 1989, 2000).

However, development of critical thinking skills in young learners is problematic (if possible at all) if teachers themselves do not think critically. In this paper we analyse a classroom activity performed by pre-service secondary school teachers attending a teacher education program. This activity aimed to develop and improve their critical thinking skills.

What is critical thinking?

There are many different definitions of critical thinking in the professional literature. In some of them critical thinking is defined as a careful, deliberate determination of whether one should accept, reject, or suspend judgment about a claim and the degree of confidence with which one accepts or rejects it(Moore andParker,1994); in others it is defined as a process or method of scientific investigation:

“Critical thinking is the process of purposeful, self-regulatory judgment" (Facione et al., 1998, p.2) in which a question is identified, a hypothesis is formulated, relevant data is sought and gathered, the hypothesis is logically tested and evaluated, and reliable conclusions are drawn from the results (Schafersman, 1991).

In other definitions critical thinking has been defined as the formation of logical inferences (Simon and Kaplan, 1989), developing careful and logical reasoning (Stahl and Stahl, 1991), and deciding what action to take or what to believe through reasonable reflective thinking (Ennis, 1991). Amidst different foci in these definitions, various national commissions have emphasized the importance of developing critical-thinking (AAAS 1989), and most educators believe that critical thinking should be the main objective of college education (Yuretich, 2004).

Rather than defining what critical thinking is, several researchers describe its components and features. For example, Pascarella and Terenzini (1991) compiled some characteristics of the ability to think critically: “ . . . critical thinking typically involves the individual’s ability to do some or all of the following: identify central issues and assumptions in an argument, recognize important relationships, make correct inferences from data, deduce conclusions from information or data provided, interpret whether conclusions are warranted on the basis of the data given, and evaluate evidence or authority” (p. 118).

In a similar way Beyer (1987) compiled a list of critical thinking skills which includes: (1) Distinguishing verifiable facts from value claims, (2) Distinguishing relevant from irrelevant information, claims, and reasons, (3) Determining factual accuracy of a statement, (4) Determining credibility of a source, (5) Identifying ambiguous claims or arguments, (6) Identifying unstated assumptions, (7) Detecting bias, (8) Identifying logical fallacies, (9) Recognizing logical inconsistencies in a line of reasoning, and (10) Determining the strength of an argument or claim.

This list is, of course, incomplete, but it serves to indicate the type of thinking and approach to real-life situations that require a good critical thinker to be: (a) systematic, (b) inquisitive, (c) judicious, (d) truth-seeking, (e) confident in reasoning, (f) open-minded and (g) analytical.

Regardless of the specific descriptions and characteristics attributed to critical thinking or to critical thinkers, Bailin and Siegel (2003) argue that “critical thinking is often regarded as a fundamental aim and an overriding ideal of education” (p.188).

At the same time, Ennis (1991) defines critical thinking as “reasonable reflective thinking that is focused on deciding what to believe or do” (pp. 473–474).

It is absolutely clear that all these skills should be implemented not only in math lessons but in everyday life. Swartz (1992) also states that students should not only employ critical thinking skills in class, but also be able to activate them in real-life situations and to recognize situations in which these skills should be used.

According to Van Gelder (2005) and Schafersman (1991), since humans are not naturally critical thinkers they must practice and actively engage in this skill; otherwise they will not improve. To be able to provide such practice to their students, teachers themselves must be proficient critical thinkers. Therefore, the development of critical thinking in teachers as a part of their professional development is an important goal of teacher education. Because of the apparent similarity between critical thinking and scientific thinking, it is reasonable to expect that mathematics and science courses afford a good opportunity to develop critical thinking.

Critical thinking in Mathematics Education

The development of students' advanced logical reasoning and critical thinking is one of the major goals of mathematics teaching in schools. With specific focus on mathematics, critical thinking includes such acts as "formulating hypotheses, alternative ways of viewing a problem, questions, possible solutions, and plans for investigating something" (Ennis, 1991, pp. 473–474).

NCTM's Curriculum and Evaluation Standards (1989) stressed: "A climate should be established in the classroom that places critical thinking at the heart of instruction ... To give students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervades all mathematical activity" (p. 25).Moreover, Ferrett (2002) included the following attributes (among others)of critical thinking: assessment of statements and arguments, admitting a lack of information, ability to clearly define a set of criteria for analyzing ideas, examining problems closely, and being able to reject information that is incorrect or irrelevant. Critical thinking involves evaluation of the thinking process – the reasoning that went into the conclusion we arrived at.

Applebaum and Leikin (2006) state that in order to develop students' critical thinking abilities mathematics teachers have, among other things, to be able to identify students' errors and misconceptions of different types. Using tasks that are oriented towards finding and studying the nature of errors is a crucial component in preparing mathematics teachers for their everyday work (ibid) in which they must be able to choose appropriate mathematical tasks, judge the advantages of particular representations of a mathematical concept, help students make connections between mathematical ideas, and grasp and respond to students' mathematical arguments and solutions. In all these stages a teacher activates his/her own critical thinking skills.

All of the abovementioned components can be found especially in tasks that require identifying mistakes in a presented solution, when the mistake is hidden, thought-provoking and not obvious.

The Study

Research goal and questions

The main goal of this study is to learn whether pre-service teachers exhibit their own critical thinking skills when coping with the task of finding hidden mistakes in the solution of a geometrical problem and whether classroom discussion can develop students' awareness of the importance of (1) combining such tasks in their teaching and (2) developing critical thinking skills in their students.

In particular, we address the following questions:

- How successful are pre-service secondary mathematics' teachers in solving a geometrical problem when a sketch is not provided?
- How successful are pre-service secondary mathematics' teachers in finding a mistake in their own/other students' solution?
- What are the explanations provided by pre-service mathematics' teachers to the nature of the detected mistake?

Participants and the setting

This paper analyzes a case study of a mathematical activity with a group (N=37) of pre-service secondary-school mathematics teachers (PMTs). At the time of the study all the participants had completed their coursework in mathematics and were enrolled in a problem-solving course. The participants were presented with a complex geometric problem, for whicha sketchwas not provided.

The PMTs were asked to: (1) solve the problem, (2) find an alternative way for solving the problem, and (3) reflect upon different results when different strategies were used.

This study is an instrumental case study. According to Stake (2005), an instrumental case study allows researchers to gain an insider's view of an issue or concern. An instrumental case study was chosen as a research tool in order to attain the inside view of PMTs on their own activation of critical thinking and, then, for learning their attitudes towards the importance of fostering critical thinking skills in their students.

The problem

The geometrical problem was chosen as a tool for this study because, on the one hand, it relates to some of the eight points of critical thinking presented by Appelbaum P.(1999), each one of which invites the teacher to consider how to implant critical thinking in the teaching and learning of mathematics in schools:

- *Obsess about functional relationships* - in this problem multifunctional relationships exist between algebra and geometry.
- *Problematize the "answer"* – it is actually the heart of the task: the contradiction in the attainedresult makes the situation very problematic.
- *Problematize the pedagogy* – the chosen task differs from regular classroom tasks that usually demand proving, calculating and finding something. The task presents another pedagogical tool that problematizes the situation that students are in.

On the other hand, the reasons for choosing the geometrical problem were that:

(1) It had different solutions;
(2) It was an interdisciplinary problem connecting algebra and geometry;
(3) Not all data (in this case it was a sketch) is given in the task and this can lead to some contradictions in the results;
(4)The problem invites discussion on the necessityof checking different kinds of shapes when the sketch is not given.

Below each of the four abovementioned reasons will be detailed.
Reason 1: Leikin and Levav-Waynberg (2008) suggested a notion of "solution spaces" that allows researchers to examine students'different abilities when solving problems in different ways. One of the abilities that can be evaluated in teachers is the critical skillsthey employ to compare different strategies used in solving the same problem. For example, they may answer the questions:

- *Is this solution standard or original?*
- *Does this solution require lot of computational work?*
- *Is this solution attainable for all students or only for those who already are acquainted with the specific theory?*
- *What are the theories that this solution employs?*

Among others Leikin and Levav-Waynberg (ibid)defined "expert solution spaces" as spaces of solutions that expert mathematicians can suggest to the problem. The "expert solution space" chosen for this study'sgeometrical problemcontains at least 3 different solutions. The strategies and techniques used in the solutions were well known to the students.
Such a task can be effective not only for developing students' critical thinking but can also be used as a didactical research tool for diagnosing students' difficulties.

Reason 2: The National Council of Teachers of Mathematics Curriculum and Evaluation Standards (1989) encouraged teachers to create opportunities for students to make connections within mathematics and across disciplines.At the same time, the problematic issueofstudents coping with problems combining algebra and geometry is well known. While algebra provides powerful techniques, Atiyah (2001) sees a danger that "when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning." Students should be able (though they usually are not) to recognize the close, integral relationship between algebra and geometry and be able to use this interconnectedness to solve mathematical problems. (See http://scimathmn.org/stemtc/frameworks/934c-algebra-geometric-problems.)

Reasoning (as part of critical thinking) is the main element common to algebra and geometry that teachers should emphasize in their lessons. However, coping with the difference between studying algebra and geometry is essential. If in algebra classes, students are constantly asked to show their work and justify their simplifications, often without formal connection to proof concepts or the proof process, in geometry classes, students are expected to learn how to write simple proofs. Thompson (1992) found that roughly 60 percent of pre-calculus students were successful at trigonometric-identity proofs, more than 30 percent could complete number-theory proofs dealing with divisibility, and less than 20 percent could handle indirect arguments or proofs by mathematical induction.

Usiskin (2004) put it this way, "The soul of mathematics may lie in geometry, but algebra is its heart," and, of course, one needs both a heart and a soul. Working on integrating algebra and geometry tasks can reduce the gap that exists between them.

Reason 3: Applebaum and Leikin (2006) found that even in-service mathematics teachers who had rich professional experience were unsuccessful in coping with an unrealistic problem that included some contradictions in the givens. Moreover, the teachers did not mention checking the givens in both algebraic and geometrical problems. In the follow-up discussion with teachers, the authors found that the teachers were surprised at their own inattention when checking givens in the algebraic task. However, checking givens (by teachers) in the geometrical problem was more complicated and not predictable.

Applebaum (2010) stated that activity on finding mistakes in a presented solution evokes student's critical thinking and found that pre-service teachers had difficulty in providing an explanation to contradictory results found when using different strategies.

Reason 4: When the sketch is not given the student is invited to complete it in his/her own way. It is interesting to note that people tend to prefer using symmetrical shapes, and this preference starts in childhood with students' difficulty in identifying geometrical shapes.

Clements and Sarama (2000a) mentioned that children reject triangles that are too "long", "bent over" or where "the point is not on the top". One factor that can influence students' ability to recognize a triangle is that most triangles used in different textbooks and other teaching materials are equilateral or isosceles triangles with horizontal bases (Clements and Sarama (2000a), and Hershkowitz (1990)). All this created prototypical examples in children's minds. So, not surprisingly, they may hold the misconception for many years that non-isosceles triangles are not real triangles (Clements and Sarama, 2000b). Levenson et al. (2011) also found that triangles that lacked symmetry were not always identified as triangles by children. They also found that only about 19% of children in ages 5-6 identified obtuse triangle as triangle. Burger and Shaughnessy (1986) mentioned that young children did not identify as a triangle a long and narrow triangle. In research done by Vighi (2004)it was found that 8-9 year-old students, when asked to draw a triangle, still drew mostly equilateral, or sometimes isosceles, but almost all of them drew acute triangles only. Clements and Battista (1992) state that the descriptive level of understanding (children should recognize and characterize geometrical shapes by their properties) can be achieved only in the intermediate and upper grades, or even in college.

One explanation of this phenomenon may be the simplicity of drawing equilateral and isosceles triangles and the aspiration of people for symmetry in shapes. Lack of obtuse triangles may be explained by the fact that we have many more subjects around us that resemble acute triangles than those that resemble obtuse triangles.

In a pre-research study in which65 in-service teachers were asked to draw one triangle, only 5 of them chose to draw an obtuse triangle. When the same group of teachers was asked to draw another (by their own classification) triangle, only 7 chose an obtuse triangle. This shows that the obtuse triangle is not the first choice, and often not even the second choice, when teachers are asked to draw some kind of triangle.

All the facts mentioned above lead us to the assumption that in solving geometrical problems about triangles, without being given a sketch, students will probably not take into account the case of an obtuse triangle, and this may result in some contradictions.

The research tool

The following geometrical problem was presented to the participants.
BM is a bisector of angle $B\,(M \in AC)$ in triangle ABC.
It is given that $CM = 4cm$, $MA = 2cm$, $BH = \sqrt{15}cm, BH \perp AC\ (M \in AC)$.
What are the lengths of the sides AB and BC in the ABC triangle?
(Note: no sketch was provided.)
First, we present 3 pre-service teachers'predictable solutions. The full and correct solution will be presented and discussed later.

The first possible solution (PS1):

Let's draw the ABC triangle with the following data according to the problem (See Figure 1):

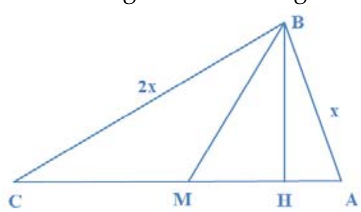

Figure 1

We know that $\frac{AB}{BC} = \frac{AM}{MC}$.
Then, it follows that $\frac{AB}{BC} = \frac{2}{4}$ (Theorem on angle bisection in a triangle).
Then I can assign $AB = x, BC = 2x$.
From ΔABH by using the Pythagorean Theorem we'll have $AH = \sqrt{x^2 - 15}$ and then
$CH = AC - AH = 6 - \sqrt{x^2 - 15}$.
From ΔBHC and using the Pythagorean Theorem we'll have the following equation:

$$\left(\sqrt{15}\right)^2 + \left(6 - \sqrt{x^2 - 15}\right)^2 = (2x)^2 \Rightarrow$$

$$15 + 36 - 12\sqrt{x^2 - 15} + x^2 - 15 = 4x^2 \Rightarrow 36 - 12\sqrt{x^2 - 15} - 3x^2 = 0 \Rightarrow 12 - x^2 = 4\sqrt{x^2 - 15} \quad (*)$$

The domain of the equation is the intersection of the following two inequalities: $12 - x^2 \geq 0$ and $x^2 - 15 \geq 0$. Solving the system of inequalities will lead us to an empty set and the conclusion: There is no triangle that fits the data presented in the problem.

The second possible solution (PS2):

Let's draw the ABC triangle with the following data according to the problem (See Figure 2):
Let's assign: $AB = x, BC = 2x$ and $MH = y$.

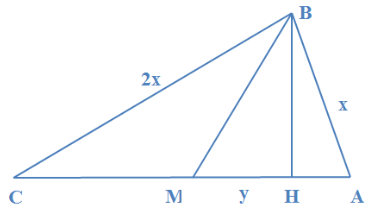

Figure 2

Then, we have asystem of two equations with two unknowns:

$$\begin{cases}(4+y)^2+\left(\sqrt{15}\right)^2=(2x)^2\\(2-y)^2+\left(\sqrt{15}\right)^2=(x)^2\end{cases}\Rightarrow\begin{cases}(4+y)^2+15=4x^2\\(2-y)^2+15=x^2\end{cases}$$

And bysolving the next equation with one unknown we'll have:

$$4(2-y)^2-(4+y)^2+45=0\Rightarrow 4y^2-16y+16-16-8y-y^2+45=0$$

$$3y^2-24y+45=0\Rightarrow y_1=3,\quad y_2=5 \text{ and } x_1=4,\quad x_2=2\sqrt{6}$$

Substitutionof these solutions (which actually is not necessary)into the system of two equations shows their correctness:

$$\begin{cases}(4+3)^2+\left(\sqrt{15}\right)^2=(2\cdot 4)^2\\(2-3)^2+\left(\sqrt{15}\right)^2=4^2\end{cases}\Rightarrow\begin{cases}7^2+15=64\\1+15=16\end{cases}\Rightarrow\begin{cases}64=64\\16=16\end{cases}$$

and

$$\begin{cases}(4+5)^2+\left(\sqrt{15}\right)^2=\left(2\cdot 2\sqrt{6}\right)^2\\(2-5)^2+\left(\sqrt{15}\right)^2=\left(2\sqrt{6}\right)^2\end{cases}\Rightarrow\begin{cases}81+15=96\\9+15=24\end{cases}\Rightarrow\begin{cases}96=96\\24=24\end{cases}$$

The third possible solution (PS 3):

Let's draw the ABC triangle with the following data according to the problem (See Figure 2):
Let's say that$AB=x$, $BC=2x$ and it's given that $AC=6cm$. According to Heron's formula for the area of a triangle:

$$S_{\Delta ABC}=\sqrt{p(p-a)(p-b)(p-c)},\quad p=\frac{a+b+c}{2}, \text{ we'll have:}$$

$$S_{\Delta ABC}=\sqrt{\frac{3x+6}{2}\left(\frac{3x+6}{2}-x\right)\left(\frac{3x+6}{2}-2x\right)\left(\frac{3x+6}{2}-6\right)}.$$

At the same time we know that $S_{\Delta ABC}=\dfrac{\sqrt{15}\cdot 6}{2}=3\sqrt{15}$.

Then we'll have: $3\sqrt{15}=\sqrt{\dfrac{3x+6}{2}\left(\dfrac{3x+6}{2}-x\right)\left(\dfrac{3x+6}{2}-2x\right)\left(\dfrac{3x+6}{2}-6\right)}$

$$12\sqrt{15}=\sqrt{(3x+6)(x+6)(6-x)(3x-6)}$$

$$4\sqrt{15}=\sqrt{\left(x^2-4\right)\left(36-x^2\right)}$$

$$240=\left(x^2-4\right)\left(36-x^2\right)$$

$$x^4-40x^2+384=0$$

Then: $x_1=2\sqrt{6}$ or $x_2=4$

Checking the roots by substitution shows that they are correct answers for the equation.

The fourth possible solution (full and correct) was dependent on a correct sketch of this problem:

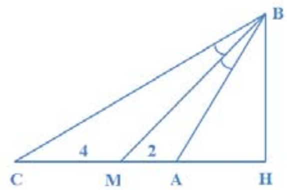

Figure 3

Then, all the previous strategies adapted to the correct sketch were also correct.

Data collection and data analysis

Research survey

There were 3 stages in the research plan.

At the first stage 37 pre-service teachers were asked to solve the problem. At the second stage they were invited to develop alternative solutions to those they had found at the first stage and compare the results that emerged from different solution paths. At the third stage the participants were guided to examine the perceived contradictions (opposite results) in each of the different solutions.

The whole-class discussions at each stage were conducted by the author and recorded by an assistant.

Stage 1:

The participants were asked to solve the problem.

All the solutions were collected by the assistant, and the results were organized in Table 1 presented below. After this we asked the pre-teachers to solve the problem in an alternative way.

Stage 2:

The participants were asked to solve the problem in an alternative way to those they had used at Stage 1. When coping with solving the problem the second time, we handed their first solutions back to them. In this way, they could see their own results from the previous stage.

Table 1. Distribution of the strategies used by pre-service teachers in coping with a geometrical task and their success rate

N=37	Number of students who used PS1 (one unknown)			Number of students who used PS2 (two unknowns)			Number of students who used PS3 (Heron's formula)		
	Solved and checked the solutions	Solved and did not check the solutions	Did not get the result (had some algebraic mistake)	Solved and checked the solutions	Solved and did not check the solutions	Did not get the result (had some algebraic mistake)	Solved and checked the solutions	Solved and did not check the solutions	Did not get the result (had some algebraic mistake)
Stage 1	2	10	3		20	2			
Stage 2	15	3	1	9	6			3	

We can learn from the collected data(Table 1) that at Stage 2 more teachers checked their results than at Stage 1.We presume that this happened because they discovered contradictory results and understood that something was wrong. Also, we can learn from the data that Heron's formula is not the first choice in teachers' set of strategies.

Stage 3:

Different solutions were presented and discussed.

(a) Dan's solution was similar usingPS1.
The gist of the discussion held in the class is captured below.

Instructor: What can you say about Dan's solution?
Inbal (checked the solution without solving it): Dan did a good job and found 2 options. They are both positive and, therefore, we have two solutions.
Instructor: How do you know that Dan's solutions are correct?
Inbal: I checked all steps of his solution and did not find a mistake.
Instructor: What do you mean by saying: "checked all steps"?
Inbal: I know that the theorem he used is correct and no algebraic mistakes were found in his solution.
Instructor: Anybody else?
Sonya: I solved this problem in the same way and actually found the same solutions. I hope I did not make the same mistake Dan probably made.
Instructor: Is it important to solve the problem (before checking)? And is it important to solve it in the same way?
Michal: I generally prefer to solve a problem before checking the correctness of its solution. Also, I think that for more effective checking it is better to follow the original solution (she means the student's solution – author.); otherwise you may get another result without realizing precisely where the student's mistake was.
Instructor: Is there anybody who found any mistakes in Dan's solution?
Meni: Yes. I checked the solution and did not find any geometrical or algebraic mistakes. In my opinion, the weak point of the solution may be only at the stage when Dan squared both sides of the equation (*).
Instructor: What do you mean by "weak point" at this stage?
Meni: We learned in our Algebra course that this action can lead us to extraneous roots.
Instructor: So?
Meni: So, one of the roots $x_1 = \sqrt{20+2\sqrt{3}}$ or $x_2 = \sqrt{20-2\sqrt{3}}$ (or both) probably appeared at this stage and are not really the equation roots. This is the reason why we have to check the final solution by substitution in the original equation or by checking the restrictions in the domain of the equation (*).
Instructor: And what did you do?
Meni: I found that the domain of the equation is the intersection of the following two inequalities: $12 - x^2 \geq 0$ and $x^2 - 15 \geq 0$. It is clear that we have an empty set.
Instructor: And your conclusion is?
Meni: None of the roots is acceptable.
Instructor: And what does it mean in our case?
Noam: Really, extra roots appeared, but where did the correct roots disappear to?
Ahava: Right, we know that extraneous roots may appear, but they do not come instead of the correct roots.
Instructor: So, Meni, what would be your final conclusion?
Meni: It seems to me that a triangle with such conditions does not exist.

At this stage, the discussion revealed some elements of critical thinking skills demonstrated by the participants. Inbal focused on the correctness of the theorem used by Dan, checked all the algebraic calculations and considered the necessity of positive values of roots. Sonya and Michal concentrated on checking the correctness of Dan's solution by solving the problem in the same way Dan did, whereas Inbal and Menimerely checked Dan's solution without solving the problem by themselves. Meni seemed to display more characteristics of a good critical thinker, as he recognized important relationships, deduced conclusions from the information or the data provided, interpreted whether conclusions are warranted on the basis of the data given, and evaluated evidence or authority. He looked for what he described as the "weak point" of the solution and found it by using his knowledge of solving radical equations in algebra. He also demonstrated flexibility in moving from algebra to geometry and back. Then he drew conclusions about the non-existence of such a triangle.

Considering alternative solutions, comparing results and conclusions

At this stage students were asked to solve the problem in an alternative way.

(b) Kobi's solution was similarusingPS2

Kobi: I solved the problem in an alternative way and found the correct roots.
Instructor: Could you tell us how you solved it?

Kobi presented the solution below, depicted in Figure 2.

After Kobihad presented his solution, the following discussion took place:

Instructor: What do you think of Kobi's solution?
Igal: It seems to be correct.
Michal: Yes, it is a good solution and he checked the roots by substitution.
Habes: Wait a minute; don't we have some contradiction here?
Instructor: What kind of contradiction?
Habes: Is it possible that using different strategies led us to opposite results?In the first case we decided that such a triangle did not exist and in the second case we stated that it did!
Instructor: What do other students think about this issue?
Alex: Right. Different correct strategies have to lead to the same result.

Other students shared Alex's opinion.

Instructor: OK. If this is so, whose solution is correct?

After 5-minute of thinking Habes suggested:

Habes: Kobi did check the roots by substitution but he did not check the existence of a triangle with the sides that he had found.
Then Habes drew the picture in Figure 3on the blackboard:
Habes: It is clear that such a triangle does not exist! If *AB=4cm* and *AM=2cm*, then *MH* has to be a part of *AM*. But we got *MH=3cm*. Then Dan's solution was correct and such a triangle does not really exist. The same situation will happen if we check the other root: $x_2 = 2\sqrt{6}$.
Instructor: Now we have seen that different strategies can lead to different numerical results.
In Dan's solution we had:

$$AB = \sqrt{20+2\sqrt{3}}cm \text{ and } BC = 2\sqrt{20+2\sqrt{3}}cm$$

or

$$AB = \sqrt{20-2\sqrt{3}}cm \text{ and } BC = 2\sqrt{20-2\sqrt{3}}cm.$$

In Kobi's solution we had:
AB=4cm and *BC=8cm*
or
$AB = 2\sqrt{6}cm$ and $BC = 4\sqrt{6}cm$.
Do we still have any contradiction here?
Soha: Yes! It seems to me that different correct strategies should not lead to different results.
Michal: Right, but in both cases we had the same final result: "Such a triangle does not exist". Hence, I don't see any contradiction here.
Instructor: Did anyone use an alternative strategy and get an alternative, or the same, result?
Suad: I have another solution, but with the same result.
Instructor: Share your solution with us, please.

(c) Suad's solution was similar using PS3.

Suad: Checking the roots by substitution shows that these are correct answers tothe equation, but we have already checked these solutions and discovered that the triangle with these sides' lengths does not exist.
Inbal: Why did we need an alternative solution? If two different ways led us to the same result ("A triangle with such parameters does not exist."), is it not enough to conclude that such a triangle does not exist?
Instructor: Your statement makes sense,but we do not yet know why a triangle with these conditions does not exist.
Katty: Maybe some data of triangle sides comes in contradiction with the giventriangle?
Instructor: Could you explain what exactly occurs with the triangle's conditions?
Katty: I don't know.

At this stage, the discussion that was led by Habes focused on the contradiction found by the students. Soha's statement: "Different correct strategies should not lead to different results" was related to recognizing logical inconsistencies and identifying logical fallacies. Habes succeeded in analyzing the data and examining the problem closely, and he was able to reject the information that was incorrect and to summarize the correct conclusion. Inbal and Katty, according to their questions, appeared to be thinkers who attempted to decide what action to take or what to believe through reasonable reflective analysis.

Does the triangle exist?

At this stage the instructor assumed a more active role in facilitating the discussion.

Instructor: Now imagine that you did not reach the conclusion that "such a triangle does not exist" at the previous stage. You use Heron's formula, x is positive, and the triangle with the sides $2\sqrt{6}cm,\ 4\sqrt{6}cm\ \ and\ \ 6cm$ does exist.

And the triangle with the sides $4cm,\ 8cm\ \ and\ \ 6cm$ exists also. Now, where do you see the contradiction? And should we check every time we have geometrical problems whether such a shape exists or not?
Suad: I don't see any contradiction. Maybe we need to analyze the properties of a triangle with sides $4cm,\ 8cm\ \ and\ \ 6cm$?
Sami: Oh, I know, we have an obtuse triangle! We know that if the following inequality exists in a triangle: $c^2 > a^2 + b^2$, then it is an obtuse triangle. And in both cases there are the following inequalities: $8^2 > 6^2 + 4^2$ and $\left(4\sqrt{6}\right)^2 > \left(2\sqrt{6}\right)^2 + 4^2$
Instructor: Let's check what the difference is in this case.
{10 minutes later…}
Amar: I checked and discovered that in both cases:
$2\sqrt{6}cm,\ 4\sqrt{6}cm,\ 6cm$ and $4cm,\ 8cm,\ 6cm$
an obtuse triangle with such conditions does exist.
Amar presented the diagram in Figure 3.
Instructor: Who wants to conclude what have we learned in this session?
Ruth: We always have to check the solutions.
Abed: We have learned that if we are dealing with some object that does not exist, using different strategies can lead us to different results.
Noam: We have to check not only the algebraic mistakes but the existence of such a shape as well.
Instructor: What do you think about this task?
Inbal: We usually solve geometrical problems with a picture given. Then we do not need to analyze whether such a triangle exists.
Suad: I always remember to check the answers in irrational equations, but in geometry it is the first time I have encountered this.
Amar: Heron's formula for the triangle's area is universal!
Noam: It was a good task for encouraging our critical reasoning and thinking.
Dan: We can formulate the next statement: "If different strategies lead to different results of the same problem, then there is probably a flaw or inaccuracy in the givens".

At this stage we found a number of statements made by pre-service teachers that demonstrate elements of critical thinking skills: suspending judgment in the absence of sufficient evidence to support a decision, attempting to anticipate probable consequences of alternative actions, applying problem-solving techniques in domains other than those in which they were learned.

Concluding Remarks

Mathematics teachers in their professional life have to deal with many different challenges. Some are related to mathematics, while others are ofa pedagogical nature. Assessing students' solutions, finding possible incoherence or errors and using them to guide students in their learning paths requires competencies that belong to the interface of mathematical and pedagogical skills. Such competencies require teachers to mobilize their own critical thinking, as well as their mathematical knowledge, pedagogical sensitivity, intellectual endurance and patience.

This paper presents a case study on the implementation of one geometrical problem by a group of pre-service secondary mathematics teachers. The activity was designed to promote and support participants' critical thinking; it included (1) solving a geometrical problem, (2) solving the problem in a different way, and (3) comparing different solutions and the resulting conclusions. The whole-class discussion showed the pre-service teachers' awareness (as a group) of possible flaws in mathematical reasoning and metacognitive mechanisms for detecting errors in different solutions. However, the instructor's guidance was crucial in helping participants discover their critical thinking skills, rather than accept seemingly correct algebraic manipulations as a reliable solution.

At the first stage we found that each participant individually held some critical thinking skills (Inbal with checking all steps in the solution, Ahava and Meni with their conclusion about extraneous roots) while as a group the participants complemented each other in terms of characteristics of good critical thinkers (Ferrett, 2002). Some of the pre-service teachers connected the knowledge of algebra and geometry which they had attained in different courses (for instance, Meni with his “weak point” when squaring both sides of the equation and then performing substitution of accepted roots in the original equation and finding the domain). All of these facts show that whole-group discussions on such kinds of tasks enrich all participants.

At the second and third stages the participants showed a variety of mathematical reasoning abilities (Habes, Soha and Alex with the statement that “Different correct strategies have to lead to the same result.”) when solving the given geometrical problem using different strategies, which, surprisingly for them, led to results inconsistent with those that they had anticipated at the first stage. During the discussion some pre-service teachers could not decide which conclusion was correct, and they experienced initial difficulties in explaining contradictions that appeared in different solutions. However, when the participants compared their findings they discovered that (a) using different strategies in problem solving could help them in checking their solutions, (b) checking the existence of all conditions when solving a problem isa necessary component in all mathematical activity, and (c) when a sketch is not provided in geometrical problems, the examination of all possible situations is required. By summarizing the discussion, the pre-service teachers agreed that this kind of mathematical activity can develop and promote their own critical thinking and that of their pupils.

Reflecting on all stages of the study, we found that the pre-service teachers experienced difficulties in seeking explanations to the apparent contradictions they faced in different solutions. However, the careful guidance of the instructor led them towards the successful completion of the task. This finding highlights the significance of the instructor’s role in pre-service teachers’ education, in general, and in the development of their critical thinking skills, in particular.

Teaching by using such kinds of problems seems to be a promising approach for the development of teachers’ critical thinking, which is a necessary prerequisite for developing and supporting critical thinking skills in their students.

The didactic questions are still open: Is using such problems legitimate in school mathematics? Should we use geometrical problems without providing a sketch? Do we expect our students to check all possible cases when a sketch is not given? Is this a good example for generating critical thinking?

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